Lecture 7: main exercises

Exercise 7.1. Let $R = \mathbb{C}[x, y]$, let $I = (x, y) \subset R$, and let $i_0 : \{0\} \hookrightarrow \mathbb{C}^2$. For each $n \geq 0$ let

$$S_n := i_{0*}(V_n),$$

where V_n is the simple SL_2-representation $\operatorname{Sym}^n \mathbb{C}^2$. For example, S_0 corresponds to the R-module R/I.

Recall from lecture that the simple objects in $\mathcal{P}^{SL_2}_{coh}(\mathbb{C}^2)$ are $R[1]$ and $\{S_n\}_{n\in\mathbb{N}}$.

(a) Show that $I^2[1]$ is perverse by identifying a composition series for it, and find the associated composition factors. Hint: use your knowledge of the simples, and the fact that $\mathcal{P}^{SL_2}_{coh}(\mathbb{C}^2)$ is closed under extensions.

(b) We indirectly showed that $I[1]$ perverse. Show this directly by explicitly computing the cohomology of $\mathbb{D}(I[1])$ for the standard t-structure, and checking that $\mathbb{D}(I[1]) \in D^b \operatorname{Coh}^{SL_2}(\mathbb{C}^2)^{\leq 0}_{p}$. Hint: I has a resolution $R \to R^2$. Hint: The resolution has differential $1 \mapsto (y, -x)$, making it the naive truncation of the Koszul resolution of R/I.

(c) Since $\mathbb{D}$ preserves $\mathcal{P}^{SL_2}_{coh}(\mathbb{C}^2)$ it follows that $\mathbb{D}(I[1])$ is also perverse. Identify a composition series for it and find the associated composition factors. Hint: $\mathbb{D}$ is a contravariant autoequivalence, hence it takes an exact triangle $X \to Y \to Z$ to an exact triangle $\mathbb{D}(Z) \to \mathbb{D}(Y) \to \mathbb{D}(X)$.

Exercise 7.2. Let $R = \mathbb{C}[x^2, xy, y^2] \subset \mathbb{C}[x, y]$, and let $X = \operatorname{Spec} R$. We have $\omega_X \cong \mathcal{O}_X[2]$. Remarks about X can be found after the exercise.

(a) Compute the stabilizer in SL_2 of a point in $U = X \smallsetminus \{0\}$, and show that it has a nontrivial one-dimensional representation.

(b) Let $M \subset \mathbb{C}[x, y]$ be the R-submodule generated by x and y. That is, M is spanned by monomials $x^a y^b$ with $a + b$ odd. Find a projective resolution of M. Hint: you can find a resolution where $P^k \cong R^2$ for all $k \leq 0$, and where d^k is the same for all $k < 0$.

(c) Use your resolution to show that $\mathbb{D}(M) \cong M[2]$, hence that $M[1]$ is perverse.

Remark 7.1. In fact, $M[1]$ is the simple corresponding to the stabilizer representation from (a), and the complete list of simples is $M[1]$, $R[1]$, and $\{S_n\}_{n\in\mathbb{N}}$ (defined as before). Note that M not locally free, even though its restriction to U is.

Remark 7.2. The space X is the quotient of $\mathbb{C}^2$ under multiplication by ± 1, relating to the fact that R is the subring of polynomials on $\mathbb{C}^2$ invariant under this action. Moreover, X is the *nilpotent cone* of $\mathfrak{sl}_2$, and can be identified with an open subset of $\mathsf{Gr}_{\leq 2\omega_1^\vee}$ containing its singular point.

Lecture 7: additional exercises

Exercise 7.3. Let $\sqcup_I U_i \to M$ be an atlas of a manifold M, i.e. each restriction $U_i \to M$ is the inclusion of an open subset homeomorphic to an open ball in $\mathbf{R}^n$, and I is a set indexing these subsets. Also let $U_{ij} := U_i \cap U_j$, $U_{ijk} := U_i \cap U_j \cap U_k$ for all $i, j, k \in I$. There is correspondence between sheaves on M and the data of (i) a sheaf $\mathcal{F}_i$ on each U_i and (ii) an isomorphism $\varphi_{ji} : \mathcal{F}_i|_{U_{ij}} \to \mathcal{F}_j|_{U_{ij}}$ for each $i, j \in I$, which together satisfy the condition (iii) $\varphi_{ki}|_{U_{ijk}} = \varphi_{kj}|_{U_{ijk}} \circ \varphi_{ji}|_{U_{ijk}}$ (i.e. as morphsms $\mathcal{F}_i|_{U_{ijk}} \to \mathcal{F}_k|_{U_{ijk}}$). Under this correspondence a sheaf $\mathcal{F}$ on M is taken to its restrictions $\mathcal{F}|_{U_i}$, and conversely we can could define the category of sheaves as the category of gluing data (i) and (ii) satisfying (iii). The various restriction maps involved in this correspondence can be gathered into a diagram

$$\bigsqcup_{I^3} U_{ijk} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \bigsqcup_{I^2} U_{ij} \rightrightarrows \bigsqcup_{I} U_i.$$

Now suppose X is a variety with an action of an algebraic group G. Write down a natural diagram

$$G \times G \times X \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} G \times X \rightrightarrows X,$$

and propose a general definition for equivariant sheaves, modeled on the definition of sheaves on M in terms of their restrictions to an atlas. Intuitively, we think of G-equivariant sheaves on X as ordinary sheaves on an object "X/G" for which X plays the role of an atlas.