WARTHOG

$L \subset S^3$ oriented link

Jones: $V_L(q) \in \mathbb{Z}[q^{\pm 1}]$

$V(O) = q + q^{-1}$

$q^2 V(\nearrow\!\!\!\!\nwarrow) - q^{-2} V(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) V(\uparrow\uparrow)$

HOMFLY-PT: $P_L(a, q)$

$aP(\nearrow\!\!\!\!\nwarrow) - a^{-1} P(\nwarrow\!\!\!\!\nearrow) = (q - q^{-1}) P(\uparrow\uparrow)$

$\overline{P}(O) = \dfrac{a - a^{-1}}{q - q^{-1}}$

Khovanov: $Kh^{i,j}(L)$ bigraded abelian group

$\chi(Kh^{i,j}(L)) = \sum (-1)^i q^j \dim Kh^{i,j}(L)$
$= V(L)$

Kh-Rozansky: $\overline{HHH}(L)$ triply graded abelian group:

$\chi(\overline{HHH}(L)) = P_L(a, q)$

Global Problem: Understand structure + geometric meaning of HHH.

Structure:

Reduced Homology: K a knot

$\Rightarrow \overline{P}_K(a,q) = \dfrac{a - a^{-1}}{q - q^{-1}} P_K(a,q) \quad \overline{P}_K \in \mathbb{Z}[a^{\pm 1}, q^{\pm 1}]$

$$HHH(K) \simeq H^*(S^1) \otimes \mathbb{Z}[x] \otimes \overline{HHH}(K)$$

$\uparrow$ f.g. over $\mathbb{Z}$.

Specialization: $P_L(q^2, q) = V_L(q)$

$P_L(q^N, q) = P_N(K) = sl_N$ knot polynomial.

$\uparrow$ Also categorified by K-R

Spectral sequence: $\overline{HHH}(L) \rightsquigarrow H_N(K)$

$\overline{HHH}(K) \rightsquigarrow \overline{H}_N(K)$

$\overline{H}_1(K) = \mathbb{Z}$.

Symmetry (??): $\overline{P}_K(a,q) = \overline{P}_K(a,q^{-1})$

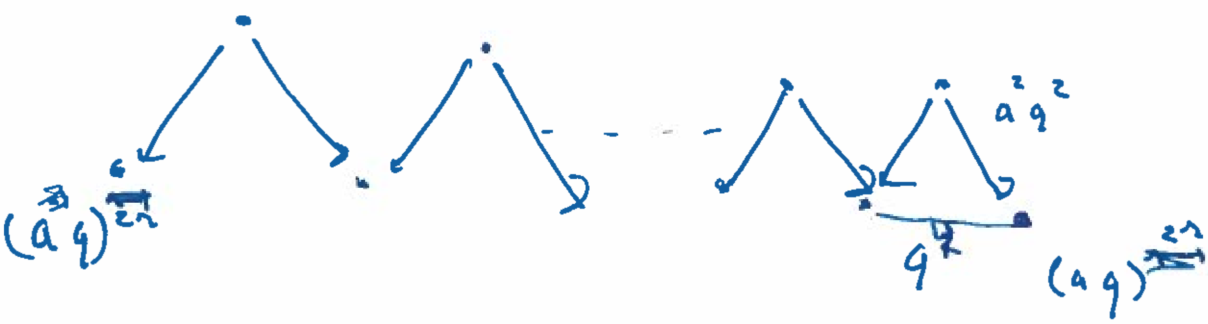
Is there an analogous symmetry for HHH?

Local Question: What is $HHH(T(n,m))$?

$T(3,4)$ 

Ex: $\overline{HHH}(T(2,5))$

$\overline{HHH}(T(2,\cancel{5}\ 2n+1))$



Ways to think about $H_{bot}(T(2,2n+1))$

① Representation Theory

$V = \mathbb{C}^2$ = vector rep'n of $G = SL_2(\mathbb{C})$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$. $V_n = \mathrm{Sym}^n V = \langle x^n, x^{n-1}y, \dots, y^n \rangle$

$T = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \subset G$. Powers of $q \Leftrightarrow$ weight of T action.

② Algebraic Geometry:

$T \circlearrowright \mathcal{O}(n)$ $\qquad V_n = \Gamma(\mathcal{O}(n))$

$\downarrow$

$T \circlearrowright \mathbb{P}^1$

$u[x;y] = [ux; u^{-1}y]$

③ Formula:

$$\frac{Q^{N+1}}{Q-T} \qquad \frac{-T^{N+1}}{Q-T}$$

$$\mathcal{P}(H_{bot}) = \frac{(qt)^{2(n+1)}}{q^2 - q^{-2}} - \frac{(qt)^{-2(n+1)}}{q^2 - q^{-2}}$$

$$(*) \quad \cancel{= \sum_{\substack{\text{fixed pts} \\ \text{of } T \text{ action}}} (\quad)} = \chi_T(\mathcal{O}(n)) = \sum_{\substack{\text{fixed pts} \\ \text{of } T \text{ action}}} (\sim).$$

What if $n < 0$?

$(*)$ doesn't make sense

Solution: ~~Replace~~

$H_{bot} = H^*(\mathcal{O}(n))$

Adding a full twist:

$T(2,2n+1) \longrightarrow T(2,2n+3)$

$\mathcal{O}(n) \longrightarrow \mathcal{O}(\ldots)$

$\mathcal{O}(n+1) = \mathcal{O}(n) \otimes \mathcal{O}(1)$

Q: How is $HHH(\overline{\tau\sigma})$ related to $HHH(\overline{\sigma})$?

$H_{top}(T(2,2n+3)) \simeq H_{bot}(T(2,2n+3))$

Thm (Kalman): $P_{top}(\overline{\tau\sigma}) \simeq P_{bot}(\overline{\sigma})$

Q: Is $HHH_{top}(\overline{\tau\sigma}) \simeq HHH_{bot}(\overline{\sigma})$?

Top Row / Bottom Row

What Happens for $T(n,m)$?

Conj: $V_n \longrightarrow L_{m/n}$ (rep of rational Cherednik algebra)

$\mathbb{P}^1 \longrightarrow Hilb^n(\mathbb{C}^2)$

$$\frac{(qt)^{n+1}-(qt)^{-(n+1)}}{q^2-q^{-2}} \longrightarrow \sum_{SYT} (\sim).$$

For $\overline{\sigma}$, $\sigma \in Br_n$

$\overline{\sigma} \longrightarrow S(\sigma)$

Lecture 2

Goal: Define HHH and H_{sl_n}

$L = \overline{\sigma}$ = braid closure:

Khovanov:

Cube of resolutions

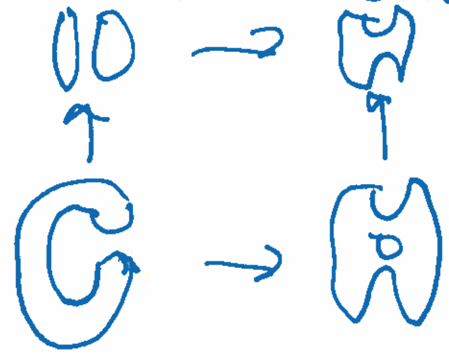


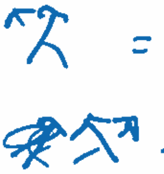
diagram $D_v \to$ group $A(D_v)$

$CKh(D) = \bigoplus_v A(D_v)$

Kauffman state model:

$\mathrm{gr\,dim}\, A(D_v) = (q+q^{-1})^{\#(D_v)}$

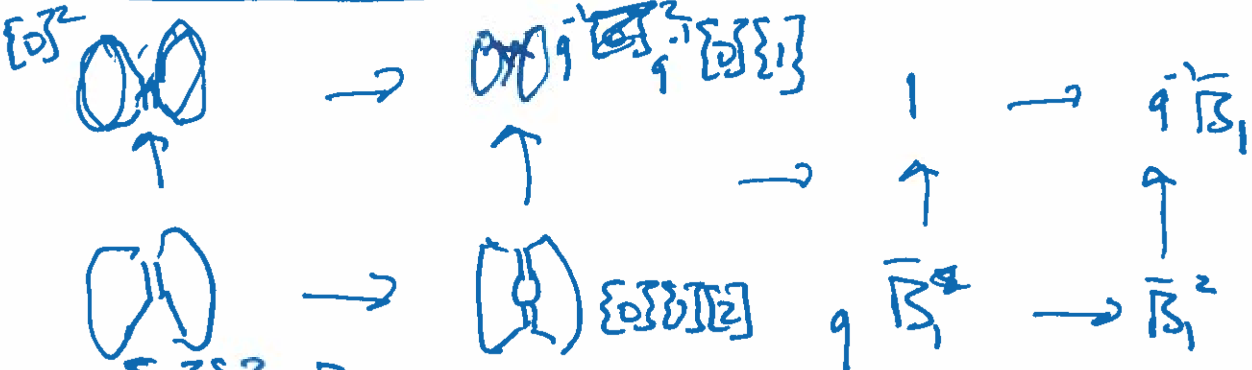
MOY state model:



$= (aq^{-1}) \cdot (q)(\;) \to)(\;)$

$= (aq^{-1})^{-1} \cdot ()(\; \to q^{-1})(\;)$

Cube of resolutions:



$\{0\}^2$ $\quad q^{-1}\{0\}\{1\}$ $\quad 1 \to q^{-1}\overline{B}_1$

$\{0\}\{1\}[2]$ $\quad q\overline{B}_1 \to \overline{B}_1^2$

$q\{0\}\{1\}$ B-S

$\underline{\mathrm{gr\,dim}}$ A(diagram) determined by MOY rules.

$\alpha \qquad \mathcal{L}(B)$

0) $\langle \overline{\mathcal{L}(B)} \rangle = \{0\} \langle \overline{B} \rangle$

I) $\langle \overline{\mathcal{L}(B)\sigma_{n-1}} \rangle = \{1\} \langle \overline{B} \rangle$

II) $\langle B_i^2 \rangle = [2] \langle B_i \rangle$

III) $\langle B_i B_{i+1} B_i \rangle - \langle B_i \rangle = \langle B_{i+1} B_i B_{i+1} \rangle - \langle B_{i+1} \rangle$

Categorification: Assign graded groups

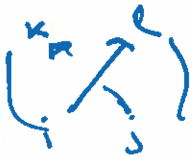
$$A(\bar{\Gamma}) \qquad \operatorname{grdim} A(\bar{\Gamma}) = \langle \bar{\Gamma} \rangle$$

Maps $A(\text{smoothing}) \to q^{-1} A(\text{singular})$

$q\, A(\text{singular}) \to A(\text{smoothing})$.

Label edges of D: $1, \dots, n$

$R = \mathbb{C}[x_1, \dots x_n]$

crossings: $C($  $) \Rightarrow$

$$\begin{tikzcd} R \arrow[r, "1"] & R \\ R \arrow[u, "(x_i - x_j)(x_i - x_k)"] \arrow[r, "(x_j - x_k)"] & R \arrow[u, "(x_i - x_k)"'] \end{tikzcd} \otimes \begin{tikzcd} R \\ R \arrow[u, "(i+j-k-l)"] \end{tikzcd}$$

$\text{singular} \to \text{smoothing}$

$C($  $)$

$$\begin{tikzcd} R \arrow[r, "1"] & R \\ R \arrow[u, "(i-k)"] \arrow[r, "(i-l)"] & R \arrow[u, "(i-j)(i-l)"'] \end{tikzcd} \otimes \begin{tikzcd} R \\ R \arrow[u, "(i+j-k-l)"] \end{tikzcd}$$

~~$C(D) \otimes_{\text{crossings of } D} C(c)$~~

$$C(D) = \bigotimes_{\text{crossings of } D} C(c) = \text{giant double complex over } R(D).$$

$A(\bar{\Gamma}) = H(C(\bar{\Gamma}), d_{int})$ $\qquad C(\bar{\Gamma}) = \text{koszul cx.}$

$HHH(D) = H(H(C(\bar{\Gamma}), d_{int}), d_{ext})$.

Problem: Why is $A(\ldots) = A(\ldots)$?

Lemma: (Koszul elimination):

$$\begin{matrix} R[y] \xrightarrow{y-a} R[y] & \sim & \\ \sim & & \text{in } K^b(R\text{-mod}) \\ 0 \to R[y]/(y-a) & & \end{matrix}$$

$$= \begin{matrix} R \xrightarrow{i-k} R \\ \otimes \\ R \xrightarrow{i+j-k-l} R \end{matrix} \sim \begin{matrix} R' \\ \otimes_R \\ R \xrightarrow{i+j-k-l} R \end{matrix} \sim R' \xrightarrow{j-l} R' \sim R'' = R/\begin{smallmatrix} i=k \\ j=l \end{smallmatrix}$$

$R' = R/(i-j)$

Eliminate linear cxs in $\otimes$ and work over $\bar{R} = R/\langle r_2 \rangle$

$r_2 = i+j-k-l$. In $\bar{R}$ $ij - kl = (i-k)(i-l) = (j-k)(j-l)$

Examples: MOY 0) $= \mathbb{C}[x_1] \xrightarrow{1-1=0} \mathbb{C}[x_1] \cdot \begin{matrix} a & \mathbb{C}[x] \\ a' & \mathbb{C}[x] \end{matrix}$

MOY I) $= \mathbb{C}[x_1, x_2]_{q^{-1}} \xrightarrow{(1-1)(1-2)} \mathbb{C}[x_1, x_2]$

$= \begin{pmatrix} \mathbb{C}[w] & \\ & \mathbb{C}[w]_{q^{-1}} \end{pmatrix} \otimes \mathbb{C}[x_1] \otimes H^*(S^1)$, $w = x_1 - x_2$

RI) $= q \to$

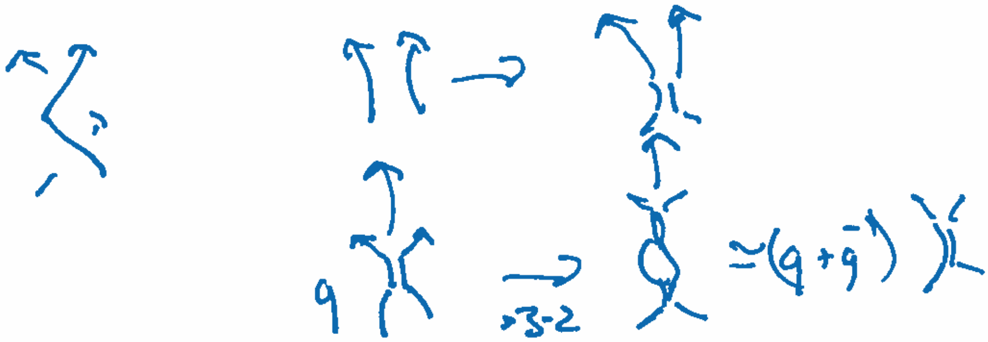
$= q \begin{pmatrix} \mathbb{C}[w] & \\ & \mathbb{C}[w] \end{pmatrix} \xrightarrow[\cdot w]{1} \begin{pmatrix} \mathbb{C}[w] \\ \mathbb{C}[w] \end{pmatrix} \otimes$

MOY II :

$$\to \quad \begin{matrix} \overline{R}[\cdot] \xrightarrow{(3-1)(3-2)} \overline{R}[3] \\ \oplus \\ \overline{R}[3] \xrightarrow{(3-5)(3-6)} \overline{R} \end{matrix} \quad = \quad \begin{matrix} \overline{R} \xrightarrow{[illegible]+12} \overline{R} \\ \oplus \\ \overline{R} \xrightarrow{4[illegible]+56} R \end{matrix}$$

$$P \sim \begin{matrix} \overline{R}/(\cdot+12) \\ \oplus \\ \overline{R} \xrightarrow{4\cdot+56} \overline{R} \end{matrix} \sim R' \xrightarrow{12-56} R' \sim (q\oplus q^{-1})\ \overline{R} \to R$$

RII)



sl_n homology:

Given $p \in \mathbb{C}[u]$ define an extra diff'l on $C(\bar{B})$ as follows:

$$R \underset{\alpha}{\overset{(x_1-x_3)(1-4)}{\rightleftarrows}} R \qquad \alpha = \frac{p(1)+p(2)-p(3)-p(4)}{(1-3)(1-4)}$$

$$(d_{int}+d_p)\Big|^2_{C(\bar{B})} = \sum_{i \text{ inner arc}} p(x_i) - \sum_{\substack{j \text{ outgoing} \\ \text{edge}}} p(x_j) = 0 \text{ on closed diagrams}$$

$$A_p(\bar{B}) = H(C(\bar{B}), d_{int}+d_p)$$
$$= (H(C(\bar{B}), d_{int}), d_p^*) \quad \text{supported in bottom row.}$$

$\Rightarrow$ spectral sequence $HHH(\bar{\sigma}) \to H_p(\bar{\sigma})$

$p = \frac{x^{n+1}}{n+1} \to sl_n$ homology.

Ex:  $\mathbb{C}[x_1] \xrightarrow{\cancel{\quad}} \mathbb{C}[x_1]$

 $= \mathbb{C}[x_i, x_j] \underset{\frac{p(i)-p(j)}{i-j}}{\overset{i-j}{\rightleftarrows}} \mathbb{C}[x_i, x_j]$ / $i = j$

 $= \mathbb{C}[x_i] \underset{p'(x)}{\overset{0}{\rightleftarrows}} \mathbb{C}[x_i]$