Lecture 5

Goal: Formulas for action of full twist.

S_n:

S_n acts on $\mathbb{C}[S_n]$ by left multiplication.

irreps V_λ $\quad \lambda \vdash n$ $\quad \dim V_\lambda = n_\lambda =$

$$\mathbb{C}_A[S_n] \simeq \bigoplus_\lambda V_\lambda^{n_\lambda} \quad \text{as rep's.}$$

As an algebra, $\mathbb{C}[S_n] \simeq \bigoplus_\lambda M_{n_\lambda \times n_\lambda}(\mathbb{C})$

ith copy of $V_\lambda = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & a & 0 \end{pmatrix}$

Hecke algebra H_n:

$$Br_n \to H_n \to S_n$$
$$\sigma_i \to T_i \to s_i$$

H_n acts on H_n by left multiplication

irreps V_λ^q $\quad \dim V_\lambda^q = n_\lambda$

As rep'n $H_n \simeq \bigoplus_\lambda (V_\lambda^q)^{n_\lambda}$

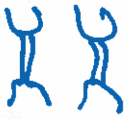
As algebra: $H_n \otimes \mathbb{C}(q) \simeq \bigoplus_\lambda M_{n_\lambda \times n_\lambda}(\mathbb{C}(q))$

Filtration + KL Basis:

$H_n = \langle C_s \rangle_{s \in S_n}$

$\pi : S_n \to \{\text{partitions of } n\}$

$s \to$ minimal partition of MOY diagram corresponding to B_s

<u>Ex</u>: $n=3$, $n=4$ 

& Partial order on partitions: $\lambda \geq \mu$ if $\sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i \ \forall k$.

$I_\lambda = \langle C_s \mid \pi(s) \geq \lambda \rangle \qquad I_{>\lambda} = \langle C_s \mid \pi(s) > \lambda \rangle$

Prop: (KL) I_λ is a 2-sided ideal in H_n.

<u>Ex</u>: $n=3$  $\langle B_{121} \rangle = I_3 \subset I_{2,1} \subset I_{1,1,1} = H_3$

<u>Ex</u>: $I_N = \langle C_s \mid \pi(s)_1 > N \rangle$ is 2-sided ideal

$TL_n = H_n / I_2$

$H_n / I_N =$ analog of TL for sl_N homology

Filtration + Algebra Structure:

Recall $H_n = \bigoplus_\lambda M_{n_\lambda \times n_\lambda}(\mathbb{Q}(q)) = \bigoplus_\lambda M_\lambda$

$I_\lambda = \bigoplus_{\mu \geq \lambda} M_\mu \qquad I_\lambda' := I_\lambda / I_{>\lambda} \xrightarrow{\cong} M_\lambda$

Ex: $M_n = \langle C_{w_0} \rangle$

Thm (KL):

$$\begin{array}{ccc} T_i : I_\lambda' & \longrightarrow & I_\lambda' \\ \big\downarrow q=1 & & \big\downarrow \\ s_i : (V_\lambda)^{n_\lambda} & \longrightarrow & (V_\lambda)^{n_\lambda} \end{array} \qquad \text{over } \mathbb{Z}[q^{\pm 1}]$$

Action of full twist:

$e_\lambda = I \in M_\lambda$ = central idempotent.

Ex: $C_{w_0} \cdot C_{w_0} = [n]! \, C_{w_0} \Rightarrow e_{(n)} = C_{w_0}/[n]!$

$T \in H_n$ is central, so $T = \sum c_\lambda e_\lambda$.

$v \in H_n \Rightarrow v = \sum e_\lambda v = \sum v_\lambda$

$T^n v = \sum c_\lambda^n v_\lambda$

$P(T^n \sigma) = \mathrm{Tr}(T^n \sigma) = \sum c_\lambda^n \mathrm{Tr}(\sigma_\lambda)$.

What is c_λ?

$P_\lambda : H_n \to M_\lambda \qquad c_\lambda^{n_\lambda} = \det P_\lambda(T)$

$= \det q P_\lambda(\sigma_1)^{n(n-1)} \qquad a_\lambda + b_\lambda = n_\lambda$

since σ_i all conjugate

$H_2 \hookrightarrow H_n$. As rep of H_2, $V_\lambda^q = (V_2)^{a_\lambda} \oplus (V_{1,1})^{b_\lambda}$

T_1 acts by 1 on $V_{1,1}$, $-q^2$ on V_2 $\Rightarrow c_\lambda = q^{2n(n-1)a_\lambda/n_\lambda}$

More idempotents:

Want to identify elements $e_T = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & 1 & & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \in M_\lambda$

e_λ is canonical; e_T are not - depend on choice of basis for M_λ.

One good way to do it:

~~As~~ $H_{n-1} \hookrightarrow H_n$. As H_{n-1} rep,

$$V_\lambda^q \simeq \underbrace{\bigoplus}_{\text{remove a box.}} V_{\lambda'}^q$$

$$c(e_{\lambda'}) = \sum_{\lambda' \subset \lambda} e_\lambda$$

$$c(e_\lambda') = \sum$$

Lecture 5

<u>Conj</u>: $HHH(T_{m/n}) = \mathrm{Hom}_{S^n}(\overset{a\text{-grading}}{\Lambda^*} \mathfrak{h}^*, L_{m/n})$ as q-graded groups.

<u>Goal</u>: where do gradings come from? diff'ls?

$d_k : HHH(T_{m/n}) \circlearrowright$ lowers a by 2, raises q by k.

How to build $\Lambda^{a-1}\mathfrak{h} \to L_{m/n}$ from $\Lambda^a \mathfrak{h} \xrightarrow{\varphi} L_{m/n}$?

As S_n reps, $\Lambda^k \mathfrak{h} \otimes \mathfrak{h}$ has unique summand $\cong \Lambda^{k-1}\mathfrak{h}$.

so given $\varphi \in \mathrm{Hom}_{S^n}(\lambda^*, H_{m/n})$, can form

$$d(\varphi) : \Lambda^{k-1}\lambda \to \Lambda^k \mathfrak{h} \otimes \mathfrak{h} \xrightarrow{\varphi \otimes d} H_{m/n} \otimes L_{m/n} \to L_{m/n}, \quad x \otimes v \mapsto xv$$

<u>Problem</u>:

$\varphi_1 : \mathfrak{h} \to$ lin polys in x_i $\to d_1$

$\varphi_{-1} : \mathfrak{h} \to$ lin polys in y_i $\quad d_{-1}$

<u>problem</u>: $[x_i, y_j] \neq 0 \Rightarrow d_{\pm 1}$ don't commute.

Solution: H_c is filtered

$$\mathbb{C}[S_n] = F_0 \subset F_1 \subset \dots \subset F_k \subset \dots \subset F_c$$

where $x_i F_j \subset F_{j+1}$
$y_i F_j \subset$

Algebra structure on associated graded is commutative.

Conj (Refined): $L_{m/n}$ is filtered:

$\cancel{L_{m/n}} = \subset g_i \subset g_{i+1} \subset \dots \subset \cancel{g}\ L_{m/n}$

$$F_i \cdot g_j \subset g_{i+j}.$$

$$HHH(T_{m/n}) \simeq \mathrm{Hom}(\Lambda^* V, \mathrm{gr}\, L_{m/n})$$

gradings determined by $Q \to q^{2} f$
$T \to q^{-2} f$

g should be constructed ~~by~~ inductively by greedy algorithm:

$$L_{m/n}^{anti} \simeq L_{m/n-1}^{sph}$$

if $x \in F_i$, $v \in g_j \subset L_{m/n-1}^{sph}$

$$xv \in g_{i+j}.$$

Stable Homology:

$$\mathcal{H}_{m/n} = \mathrm{Hom}_{S_n}(\Lambda^* V, \mathrm{gr}\, L_{m/n})$$

$L_{m/n} = M_{m/n} / I_{m/n}$ $\quad$ $I_{m/n}$ in degree m,

so $\lim\limits_{m\to\infty} \mathcal{H}_{m/n} = \mathrm{Hom}_{S_n}(\Lambda^* V, M_{\infty/n})$ exists.

$$\overline{M}_{\infty/n} = \mathbb{C}[X_1, \dots X_n] = R$$

As S_n rep, $R \simeq R_{S_n} \otimes R^{S_n}$

$R^{S_n} = \mathbb{C}[e_1, \dots e_n]$ = ring of invariants

$R_{S_n} = R/(e_1, \dots e_n)$ = ring of coinvariants.

As S_n rep, R_{S_n} is regular rep'n.

$$\Rightarrow \mathrm{Hom}(\Lambda^* V, R) \simeq \underbrace{\Lambda^*(\xi_1, \dots \xi_n) \otimes \mathbb{C}[e_1, \dots e_n]}_{\text{IS } \overline{HHH}(T^{\infty}_{\infty/n})}$$

Thm (Hogancamp):

ξ_i characterized by

$d_i \xi_i = 1 \xrightarrow{\text{symmetry}} \beta_{w_0}$ acyclic under $d_1, \dots d_i$

$$d_k(\xi_j) = \sum_{i_k = j} u_{i_1} \dots u_{i_k}$$

BGG resolution:

$I_{m|n} \subset M_{m|n}$ generated by $Y_{m|n} \simeq V \subset R$

Koszul resolution

$$\Lambda^{n-1} Y_{m|n} \otimes R \to \cdots \to \Lambda^2 Y_{m|n} \otimes R \to Y_{m|n} \otimes R \to R \to L_{m|n}.$$

is exact.

Use to compute ~~graded dim's of~~ $L_{m|n}$ as $\mathfrak{}$, hence $H_{m|n}$.

Ex: $n=2$ $\quad R = \mathbb{C}[U] = \dfrac{\ydiagram{2}}{1-q^4} + \dfrac{q^2 \ydiagram{1,1}}{1-q^4}$

$$V \otimes R = q^{2m} \ydiagram{1,1} \otimes R = \frac{q^{2n} \ydiagram{1,1}}{1-q^4} + \frac{q^{2(m+1)} \ydiagram{2}}{1-q^4}.$$

Possible ways to express the 3rd grading on HHH.

① Homological:

② t grading $t = h - a/2$

$h = t + a/2$

③ filtration grading:

$$f = \frac{q + (m-1)(a-1)}{2} + h$$