

WARTHOG 2016
Exercises for Monday morning

1. Let R be a commutative ring and $a \in \mathbb{R}$. Show that the complex $R[x] \xrightarrow{x-a} R[x]$ is homotopy equivalent to $0 \rightarrow R[x]/(x-a) \rightarrow 0$ as a complex of R -modules, but not as a complex of $R[x]$ modules.
2. Let $L = \bar{\sigma}_1^2$ be the positive Hopf link. Compute $HHH(L)$ directly from the definition: draw the cube of resolutions, identify the groups appearing at each vertex of the cube, and describe the maps between them. (You may find the proof of invariance under the Reidemeister II move helpful for the last part.)
3. Using problem 4 on the preliminary exercises to compute the reduced Khovanov homology of $T(2, n)$. Show that there is a spectral sequence with E_1 term $[2]Kh^r(L)$ which converges to $Kh(L)$. Describe this sequence for $L = T(2, n)$.
4. Recall that $HHH(\bar{\sigma})$ is the E_2 term of a spectral sequence.
 - (a) Show that the homology of the total complex depends only on the number of components of $\bar{\sigma}$. (Hint: compute the total homology using a different spectral sequence.)
 - (b) Show that the second differential in the first spectral sequence lowers the a and q gradings by 2. How does this differential act on $HHH(T(2, n))$?
5. The aim of this exercise is to understand the \mathfrak{sl}_n homology of Bott-Samelsons for $n = 1, 2$.
 - (a) Let \mathbf{B} be a Bott-Samelson diagram which is not the identity bimodule. Show that the $\mathfrak{sl}(1)$ homology of \mathbf{B} is trivial.
 - (b) If \mathbf{B} is an Bott-Samelson diagram, let $D_{\mathbf{B}}$ be the tangle diagram obtained by erasing all thick edges in \mathbf{B} . Show that the $\mathfrak{sl}(2)$ homology of the closure of B is isomorphic to $\mathcal{A}(D_{\mathbf{B}})$.

Supplementary Exercises

6. (Koszul complexes). Let V be a free R -module of rank n , and let $a \in V^*$. Show that

$$\Lambda^n(V) \xrightarrow{*a} \Lambda^{n-1}(V) \xrightarrow{*a} \cdots \xrightarrow{*a} \Lambda^1(V) \xrightarrow{*a} \Lambda^0(V)$$

is a Koszul complex (in the sense that it is a tensor product of short complexes.) By considering a change of basis for V , show that the Koszul complexes

$$\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x \\ y + ax \end{pmatrix}$$

are isomorphic. Use this to give an alternate proof that

$$\begin{pmatrix} X_1 + X_2 - X'_1 - X'_2 \\ X_1 X_2 - X'_1 X'_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X_1 + X_2 - X'_1 - X'_2 \\ (X_1 - X'_1)(X_1 - X'_2) \end{pmatrix}$$

are isomorphic.

7. We say that a category \mathcal{C} is *graded* if $\text{Mor}(A, B)$ is a graded vector space for all $A, B \in \mathcal{C}$, and $\text{gr } f \circ g = \text{gr } f + \text{gr } g$. \mathcal{C} is *well-graded* if $\text{gr } f \geq 0$ for all $f \in \text{Mor}(A, B)$ and $\text{gr } f = 0$ if and only if f is an isomorphism.

- (a) Show that $\text{Cob}(n)$ is well-graded.
- (b) Show that any complex C over a well-graded category is homotopy equivalent to a minimal complex C_{\min} in which all morphisms have positive degree.
- (c) Show that if two minimal complexes are homotopy equivalent, they are homeomorphic.