WARTHOG 2016
Exercises for Monday afternoon

1. Fill in the details in the computation of $C(\sigma_1^n)$ that we did in the lecture. (In particular, check that the morphisms are correct.) Explain how this complex is related to the complex of problem 4 on the preliminary exercises. Use it to compute $HHH(T(2,n))$.

2. This exercise is about computing Hom spaces of Bott-Samelsons.

   (a) Show that the elementary Bott-Samelson bimodule $\mathbf{B}_i$ is self-dual (as an R-R bimodule). Deduce that if $\mathbf{B}$ and $\mathbf{B}'$ are Bott-Samelsons, then

   $$\mathrm{Hom}(\mathbf{B}, \mathbf{B}') = HH^0(\mathbf{B}^r \otimes \mathbf{B}'),$$

   where $\mathbf{B}^r$ is the reverse of $\mathbf{B}$: if $\mathbf{B} = \mathbf{B}_{i_1} \cdots \mathbf{B}_{i_k}$, then $\mathbf{B}^r = \mathbf{B}_{i_k} \cdots \mathbf{B}_{i_1}$. Explain how this is related to problem 2 of the preliminary exercises.

   (b) Use the MOY rules to compute the graded dimension of $\mathrm{Hom}(\mathbf{B}_s, \mathbf{B}_t)$, where $s, t \in S^3$.

   (c) Describe the space $\mathrm{Hom}(B_{12}, B_{21})$ explicitly.

3. Consider the Rouquier complex of a positive crossing $C(\sigma)$. Show that the maps $X_1, X_2' : C(\sigma) \to C(\sigma)$ given by multiplication by X_1 and X_2' are homotopic. Deduce that $HHH(L)$ can be naturally viewed as a module over $\mathbb{Z}[X_1, \ldots, X_l]$, where l is the number of components of L.

4. This exercise is about the full twist on $n = 3$ strands.

   (a) Let $H = \sigma_1\sigma_2\sigma_1$ be the half-twist. Compute $C(H)$. (Hint: after cancellation, all relevant morphism spaces are 1-dimensional.)

   (b) By comparing with $C(\sigma_2\sigma_1\sigma_2)$, deduce that HHH is invariant under the third Reidemeister move.

   (c) Let $T = H^2$ be the full twist. Compute the minimal complex for $C(T)$ at the level of objects.

   (d) Consider the minimal form of $C(T^k)$, with the homological grading renormalized so that the lowest nonzero term in the complex is in homological grading 0. Show by induction that the only objects appearing in homological degrees $0, .., k$ are copies of $\mathbf{B}_{121}$.

5. Let $\overline{\mathbf{H}}_c$ and $\mathbf{H}_c$ be the rational Cherednik algebras associated to the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ with parameter c. Show that $\overline{\mathbf{H}}_c \simeq \mathcal{D} \otimes \mathbf{H}_c$, where D is an algebra generated by two elements $x = \frac{1}{n}\sum x_i$ and $y = \frac{1}{n}\sum y_i$ satisfying $[x, y] = 1$. Similarly, show that $\overline{M}_c \simeq M_c \otimes \mathbb{C}[x]$. Discuss how this is compatible with the relation between reduced and unreduced versions of HHH.

6. Let $\overline{\mathbf{H}}_c$ and $\mathbf{H}_c$ be the rational Cherednik algebras associated to the Lie algebras $\mathfrak{gl}_n$ and $\mathfrak{sl}_n$ with parameter c. Find explicit polynomials of degree m in the polynomial representation of $\overline{\mathbf{H}}_{m/n}$ which are annihilated by the action of the Dunkl operators when $m/n = 1/n$ and $m/n = 2/3$. (As discussed in the lecture, these generate the ideal $I_{m/n}$.)

7. The goal of this exercise is to give a hands-on proof of the symmetry of finite dimensional representations of $\mathbf{H}_c$.

   (a) Let $\overline{\mathbf{h}} = \frac{1}{2}\sum x_i y_i + y_i x_i, \overline{\mathbf{x}} = \frac{1}{2}\sum x_i^2$, and $\overline{\mathbf{y}} = \frac{1}{2}\sum y_i^2$. Show that $\overline{\mathbf{h}}, \overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ generate an action of $\mathfrak{sl}_2$ on $\mathbf{H}_c$. If p is a homogenous element of $\overline{H}_c$, show that $[\mathbf{h}, p] = (\deg p)p$.

   (b) Use the decomposition from exercise 5 to construct analogous elements $\mathbf{h}, \mathbf{x}$ and $\mathbf{y}$ generating an action of $\mathfrak{sl}_2$ on $\mathbf{H}_c$.

   (c) If v is a homogenous element of$M_{m/n}$, show that $\mathbf{h} \cdot v = (\deg v)v$, where $\deg v$ is the degree of v as a polynomial minus $(m-1)(n-1)$. (Hint: first do it for $v = 1$.)

   (d) Deduce that if $M_{m/n}$ is finite dimensional, it admits an involution ι which commutes with the action of S_n and satisfies $\deg \iota(v) = -\deg v$.

Supplementary Exercises

8. Using the formula for $[D_j, X_i]$ we derived in class, prove that $[[D_i, D_j], X_k] = 0$. Deduce that $[D_i, D_j] = 0$.

9. Let $\mathbf{B}$ be a Bott-Samelson diagram on n strands. Show that $HH(\mathbf{B})$ is supported in a-gradings $-n+1, -n+3, \ldots, n-3, n-1$. Deduce the Morton-Franks-Williams inequality: if $\sigma \in \mathrm{Br}_n$, then $HHH(\sigma)$ is supported in a gradings between $w - n + 1$ and $w + n - 1$, where w is the writhe of the braid (the number of positive crossings minus the number of negative crossings.)