WARTHOG 2016
Exercises for Tuesday afternoon

1. For each indecomposable Soergel bimodule $\mathbf{B}_s$ ($s \in S_3$), compute $T\mathbf{B}_s$. (Hint: $T = H^2$.) Verify that the degree shifts are compatible with the content formula.

2. Let $\mathbf{B}_{w_0}$ be the Soergel bimodule corresponding to the longest word in S_n. Compute $T\mathbf{B}_{w_0}$ and show that it is compatible with the content formula.

3. Using your answer to the previous exercise, compute the minimal form of $CKh(T(3,\infty)$. What are the morphisms in this complex? (Hint: use $d^2 = 0$, rather than computing directly.) Compute $Kh^r(T(3,\infty))$. Check that the homology is isomorphic to the homology of the conjectural differential d_2 on $HHH(T(3,\infty))$.

4. Let $T \subset SL_n(\mathbb{C})$ be the diagonal torus. What are the fixed points of the action of T on $\mathrm{Fl}_n(\mathbb{C})$ (the variety of complete flags in $\mathbb{C}^n$). Let L_i be the ith tautological line bundle over $\mathrm{Fl}_n(\mathbb{C})$. Compute the weights of the action of T on the fibre of L_i over the fixed points.

5. Continuing with the previous problem, let L_1 and L_2 be the tautological line bundles over $\mathrm{Fl}_3(\mathbb{C})$. Use equivariant localization to compute the equivariant Euler characteristic of $L_1^a L_2^b$ for $a, b > 0$.

6. Show that $\mathrm{Hilb}^n(\mathbb{C}^2) \simeq \mathbb{C}^2 \times X_n$, where X_n is the preimage of 0 under the composite map $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$, where the second map sends $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ to $\frac{1}{n}\sum \mathbf{v}_i$. Let $\mathcal{T}$ be the tautological bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$. Show that $\mathcal{T} \simeq \mathcal{O} \oplus \mathcal{T}'$ for some vector bundle $\mathcal{T}'$, where $\mathcal{O}$ is the trivial line bundle. What is $\mathcal{T}'$ when $n = 2$?

7. Use the matrix description of the Hilbert scheme to show that $\mathrm{Hilb}^2(\mathbb{C}^2) \simeq \mathbb{C}^2 \times E$, where E is the total space of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1$.

8. Consider the projection $\mathrm{Hilb}^3(\mathbb{C}^2) \to \mathrm{Sym}^3(\mathbb{C}^2)$. What is the preimage of a point $\{a, a, b\} \in \mathrm{Sym}^3(\mathbb{C}^2)$, where $a \neq b$?

Supplementary Exercises

9. Show there is a long exact sequence

$$\ldots \to H^*(X) \to H^{*-1}_{S^1}(X) \xrightarrow{\cdot u} H^{*+1}_{S^1}(X) \to H^{*+1}(X) \to \ldots$$

10. Let $G = GL_n(\mathbb{C})$, and let $B \subset G$ be the Borel subgroup of upper triangular matrices. Then $G/B \simeq \mathrm{Fl}_n(\mathbb{C})$. Let $R = \mathbb{C}[X_1, \ldots, X_n]$. Fill in the details of the following argument showing that $H^*(\mathrm{Fl}(n, \mathbb{C}))$ is isomorphic to the coinvariant ring R_{S_n}. $H^*_G(G/B) \simeq H^*_B(*) \simeq R$, so $H^*(G/B) \simeq \mathbb{C} \otimes_{H^*_G(*)} H^*_G(G/B) \simeq R/R^{S_n}$. Deduce that the graded dimension of the coinvariant ring is $[n]!$.