WARTHOG 2016
Exercises for Wednesday

1. Consider the following elements in H_c:

$$\mathbf{e} = \sum_i x_i^2, \mathbf{f} = \sum_i y_i^2, \mathbf{h} = \sum_i (x_i y_i + y_i x_i).$$

Prove that $\mathbf{e}, \mathbf{f}, \mathbf{h}$ span the Lie algebra $\mathfrak{sl}_2 \subset H_c$.

2. Consider the standard H_c-module $M_c(\lambda) = V_\lambda \otimes \mathbb{C}[x_1, \dots, x_n]$.

   (a) Describe the action of $\mathbf{h}$ on $V_\lambda \subset M_c(\lambda)$ (recall that $y_i(V_\lambda) = 0$).

   (b) Compute the graded character of $M_c(\lambda)$, that is, the trace of $q^{\mathbf{h}}\sigma$ for $\sigma \in S_n$.

   (c) Prove that the bigraded Hilbert series of $\mathrm{Hom}_{S_n}(\wedge^* V, M_c(\lambda))$ (graded by $\mathbf{h}$ and the exterior degree) equals $q^{\cdots}\varepsilon(s_\lambda)$, where s_λ is the Schur function and $\varepsilon(p_k) = \frac{1-a^k}{1-q^k}$. Determine the graded shift $(\dots)$.

3. (a) Describe all irreducible representations in category $\mathcal{O}$ for $H_{1/3}(S_3)$.

   (b) Find all values of c such that $D_i(W) = 0$, where W is the Vandermonde determinant $W = \prod_{i<j}(x_i - x_j)$.

   (c) Describe all irreducible representations in category $\mathcal{O}$ for $H_{1/2}(S_3)$.

4. (a) Use the Rosso-Jones formula to compute the uncolored HOMFLY polynomial for torus knots.

   (b) Recall that the finite dimensional representation $L_{m/n}$ has a BGG resolution

$$L_{m/n} \longleftarrow M_{m/n}(n) \longleftarrow M_{m/n}(n-1,1) \longleftarrow M_{m/n}(n-2,1^2) \longleftarrow \dots \longleftarrow M_{m/n}(1^n).$$

   Use problem 2 to compute the bigraded Hilbert series of

$$\mathcal{H}_{m/n} := \mathrm{Hom}_{S_n}(\wedge^* V, L_{m/n}).$$

   (c) Check that the answers in (a) and (b) agree.

5. Use the Rosso-Jones formula to compute the S^2–colored HOMFLY invariant of the trefoil.

6. * Let $f_i = \frac{\partial}{\partial x_i}\mathrm{Coef}_{m+1} \prod_{i=1}^n (1 - z x_i)^{m/n}$.

   (a) Prove that $\sum_i f_i = 0$ and hence f_i span a copy of the $(n-1)$–dimensional reflection representation of S_n.

   (b) Prove that $D_i(f_j) = 0$ for all i and j for $c = m/n$.

(c) Conclude that $L_{m/n} = \mathbb{C}[x_1, \dots, x_n]/(f_1, \dots, f_n)$.

7. In the lecture we have seen how a B_i can be moved underneath a crossing strand, which motivates the definition of a complex of singular Soergel bimodules for the $(2,1)$-crossing. Now try to slide a B_i underneath a 2-labelled strand and compare to the complex for the $(2,2)$-crossing.

8. Starting from the complex of singular Soergel bimodules for the (i,j)-crossing, try to guess the complex for the negative (j,i)-crossing. Decategorify the positive and negative crossing complexes to linear combinations of webs and use the extended MOY rules to check that the Reidemeister 2 move holds in the case of $j = 1$. Next, try to reduce the general case to the case of $(1,1)$ crossings. Use a similar trick to prove the Reidemeister 3 move for strands of any labels.

9. Use the extended MOY rules to compute the graded dimensions of the morphism spaces between consecutive terms in the complex of singular Soergel bimodules associated to an (i,j)-crossing. Argue why these complexes are essentially uniquely determined (assuming that they satisfy the Reidemeister 2 move up to homotopy equivalence).

10. (more work, but fun) Use the triangular decomposition of the skew Howe dual quantum group $U_q(\mathfrak{gl}_m)$ to explain why the extended MOY rules are sufficient to evaluate every closed braid-like web to an element of $\mathbb{Z}[a^{\pm 1}](q)$.

11. Show that $\mathrm{Hilb}^n(\mathbb{C}^2) \simeq \mathbb{C}^2 \times X_n$, where X_n is the preimage of 0 under the composite map $\mathrm{Hilb}^n(\mathbb{C}^2) \to \mathrm{Sym}^n(\mathbb{C}^2) \to \mathbb{C}^2$, where the second map sends $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ to $\frac{1}{n}\sum \mathbf{v}_i$. Let $\mathcal{T}$ be the tautological bundle on $\mathrm{Hilb}^n(\mathbb{C}^2)$. Show that $\mathcal{T} \simeq \mathcal{O} \oplus \mathcal{T}'$ for some vector bundle $\mathcal{T}'$, where $\mathcal{O}$ is the trivial line bundle. What is $\mathcal{T}'$ when $n = 2$?

12. Use the ADHM of the Hilbert scheme to show that $\mathrm{Hilb}^2(\mathbb{C}^2) \simeq \mathbb{C}^2 \times E$, where E is the total space of the line bundle $\mathcal{O}(-2)$ on $\mathbb{P}^1$.

13. Consider the projection $\mathrm{Hilb}^3(\mathbb{C}^2) \to \mathrm{Sym}^3(\mathbb{C}^2)$. What is the preimage of a point $\{a, a, b\} \in \mathrm{Sym}^3(\mathbb{C}^2)$, where $a \neq b$?