1. Review of some basic theory

In this chapter, I want to quickly run through some basic theorems which you probably remember either from 600 algebra or from Frank’s course. A good reference for a slightly more detailed review if you want it is chapter one of Benson’s book “Representations and cohomology I”.

(1) The Jordan-Hölder theorem. A composition series for an $R$-module $M$ is a (finite) series of submodules

$$0 = M_0 < M_1 < \cdots < M_n = M$$

such that each $M_i/M_{i-1}$ is irreducible (or simple). The Jordan-Hölder theorem says that any two (finite) series of submodules of an arbitrary module $M$ can always be refined to series of equal length such that the factors in one series are isomorphic to the factors in the other series, possibly up to a permutation.

(2) Chain conditions. A module $M$ satisfies DCC if every descending chain of submodules eventually stops, and ACC if every ascending chain eventually stops. By the Jordan-Hölder theorem, $M$ has a composition series if and only if it satisfies both ACC and DCC, in which case any series of submodules of $M$ can be refined to a composition series. In that case, for a simple module $L$, the composition multiplicity $[M:L]$ counting the number of factors of a composition series of $M$ that are isomorphic to $L$ is a well-defined invariant of $M$.

(3) Noetherian rings. A ring $R$ is called Noetherian if the regular module $RR$ satisfies ACC on submodules (a.k.a. left ideals). All the rings we’ll meet in this course will be Noetherian. If $R$ is Noetherian, every finitely generated $R$-module satisfies ACC. Moreover, every submodule of a finitely generated $R$-module is also finitely generated.

(4) Completely reducible modules. The socle of an $R$-module $M$ is the sum of all the irreducible submodules of $M$, written soc $M$. A module $M$ is called completely reducible (or semisimple) if $M = \text{soc } M$, which by Zorn’s lemma is equivalent to the statement that every submodule of $M$ has a complement, or to the statement that $M$ is a direct sum of irreducible modules. By its definition, the socle soc $M$ is the largest completely reducible submodule of $M$. The radical of $M$ is the intersection of all the maximal submodules of $M$, written rad $M$. If $M$ satisfies DCC, then $M$ is semisimple if and only if rad $M = 0$. So assuming that $M$ satisfies DCC, $M/\text{rad } M$ is the largest completely reducible quotient of $M$.

(5) The Jacobson radical. The Jacobson radical $J(R)$ of $R$ is the intersection of the maximal left ideals of $R$, or equivalently, the intersection of the annihilators of all the simple $R$-modules. The latter description makes it clear that $J(R)$ is a two-sided ideal of $R$. Note
tautologically that $J(R) = \text{rad}(R)$. We have Nakayama’s lemma: If $M$ is a finitely generated $R$-module and $J(R)M = M$ then $M = 0$.

(6) Artinian rings. A ring is called Artinian if $R$ satisfies DCC on left ideals. In that case, (i) every finitely generated $R$-module has DCC on submodules; (ii) by Nakayama’s lemma, $\text{rad}(M) = J(R)M$ for any finitely generated $M$; (iii) $J(R)$ is nilpotent, i.e. $J(R)^n = 0$ for some $n$. Now suppose that $M$ is a finitely generated $R$-module, and let $M_i = J(R)^i M$. Then, $\text{rad}(M_i/M_{i+1}) = J(R)(M_i/M_{i+1}) = 0$, so $M_i/M_{i+1}$ is completely reducible. Since $M$ has DCC, so does $M_i/M_{i+1}$, hence it is a finite direct sum of irreducible modules, hence it has a composition series. Since $J(R)$ is nilpotent, $M_n = 0$ for some $n$, hence $M$ itself has a composition series. This shows: if $R$ is Artinian, then every finitely generated $R$-module has a composition series. Note applying this to the regular module, you get that Artinian rings are Noetherian.

(7) Schur’s lemma. Schur’s lemma says that if $M \not \cong N$ are irreducible $R$-modules, then $\text{Hom}_R(M, N) = 0$, while $\text{End}_R(M, M)$ is a division ring. Thus, if $M$ is a finite direct sum of irreducible $R$-modules, say $M = M_1^{n_1} \oplus \cdots \oplus M_r^{n_r}$, with $M_i \not \cong M_j$, you get that $\text{End}_R(M) \cong M_{n_1}(\text{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\text{End}_R(M_r))$, a direct sum of matrix algebras over division rings. There is a stronger form of Schur’s lemma too when $R$ is an algebra over an algebraically closed field $k$ and $M$ is a finite dimensional irreducible $R$-module: in that case, $\text{End}_R(M) = k$. Proof: take $f \in \text{End}_R(M)$. Let $\lambda \in k$ be an eigenvalue. Then, $\text{ker}(f - \lambda)$ is a non-zero $R$-submodule of $M$, so its all of $M$ as $M$ is irreducible. Hence, $f = \lambda$ is a scalar and $\text{End}_R(M) = k$.

(8) Wedderburn’s theorem. A ring $R$ is called semisimple if $J(R) = 0$ (so $R/J(R)$ is always a semisimple ring!). Assuming that $R$ is Artinian, $R$ is semisimple if and only if every $R$-module is completely reducible, which is if and only if the regular module is completely reducible. So we can decompose $R = R^{n_1} \oplus \cdots \oplus R^{n_r}$ for irreducible modules $M_i \not \cong M_j$, and get by Schur’s lemma that

$$R \cong \text{End}_R(R)^{op} \cong M_{n_1}(\text{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\text{End}_R(M_r)),$$

a direct sum of matrix algebras over division rings. Moreover, $R$ has exactly $r$ different irreducible modules up to isomorphism, of dimensions $n_1, \ldots, n_r$, namely the modules of column vectors over each of the matrix algebras. Hence, the numbers $r, n_1, \ldots, n_r$ and the division rings $\text{End}_R(M_i)$ are uniquely determined by $R$. Conversely, any finite direct sum of matrix algebras over division rings is a semisimple Artinian ring.

(9) Fitting’s lemma. Suppose the $R$-module $M$ has a composition series (i.e. both ACC and DCC). Let $f \in \text{End}_R(M)$. Fitting’s lemma says that for sufficiently large $n$, $M = \text{im}(f^n) \oplus \text{ker}(f^n)$. Let me
explain the main application of Fitting’s lemma. An $R$-module is called indecomposable if it cannot be written as a direct sum of two non-zero submodules. A local ring is a ring $R$ with a unique maximal left ideal (which must therefore be equal to its Jacobson radical since that is the intersection of all the maximal left ideals). Now I claim that if $M$ is an indecomposable module having a composition series, then $\text{End}_R(M)$ is a local ring. Let $I$ be a maximal left ideal of $E = \text{End}_R(M)$. Pick $a \notin I$. We need to show that $Ea = E$. Well, $E = Ea + I$, so we can write $1_E = \lambda a + \mu$ with $\lambda \in E, \mu \in I$. By Fitting’s lemma $M = \text{im}(\mu^n) \oplus \ker(\mu^n)$ for some $n$. But $M$ is indecomposable, so either $\mu^n$ is onto or $\mu^n = 0$. The former cannot occur since $\mu$ is not a unit, so it is not an automorphism of $M$. Hence, $\mu^n = 0$. Now $(1 + \mu + \cdots + \mu^{n-1})\lambda a = (1 + \cdots + \mu^{n-1})(1 - \mu) = 1 - \mu^n = 1$, so $Ea = E$. Using this you can now easily show that if $R$ is an Artinian ring, then a finitely generated $R$-module $M$ is indecomposable if and only if $\text{End}_R(M)$ is a local ring.

(10) The Krull-Schmidt theorem. Suppose $R$ is Artinian. Then, every finitely generated $R$-module decomposes uniquely up to isomorphism as a direct sum of indecomposable modules. You can show more generally that if $R$ is any old ring and $M$ is an $R$-module that is a direct sum of finitely many indecomposables $M_i$ such that each $\text{End}_R(M_i)$ is a local ring, then all other decompositions of $M$ as a direct sum of indecomposables are isomorphic to the given one.

(11) Projective modules. Remember that a module $P$ is called projective if every map from $P$ to a quotient of a module $M$ lifts to a map from $P$ to $M$ itself. This is equivalent to the statement that every map $M \rightarrow P$ splits, and that $P$ is a summand of a free module. A projective generator means a finitely generated projective $R$-module $P$ such that every $R$-module is a quotient of a direct sum of copies of $P$.

(12) The Morita theorem. Let $R$ and $S$ be rings. The following are equivalent: (i) $R$-$\text{Mod}$ and $S$-$\text{Mod}$ are equivalent categories; (ii) $R$-$\text{mod}$ and $S$-$\text{mod}$ are equivalent categories; (iii) there exist an $R,S$-bimodule $RM_S$ and an $S,R$-bimodule $SN_R$ such that $M \otimes SN \cong R$ as an $R$-$\text{bimodule}$ and $N \otimes R M \cong S$ as an $S$-$\text{bimodule}$; (iv) there is a projective generator $P$ for $R$ such that $S \cong \text{End}_R(P)^{\text{op}}$. In that case, $R$ and $S$ are said to be Morita equivalent.

Exercise 2. (i) Prove that $\text{rad}(M \oplus N) = \text{rad} M \oplus \text{rad} N$.
(ii) Prove directly that the algebra $M_n(D)$ of $n \times n$ matrices over a division ring is a simple ring, i.e. it has no non-trivial two-sided ideals.
(iii) Deduce from (i) and (ii) that a finite direct sum of matrix algebras over division rings is a semisimple ring.

Exercise 3. Let $M$ be an $R$-module, and let $X,Y$ be submodules such that $M/X$ is semisimple and $M/Y$ is irreducible. Prove that $M/(X \cap Y)$ is
semisimple. Hence prove the statement made in (4) above that if $M$ satisfies DCC, then $M$ is semisimple if and only if $\text{rad} M = 0$.

Let me now give a few more basic examples.

**Example 1.1. Division rings.** Let $D$ be a division ring (e.g. a field!). Since $D$ is a simple $D$-module, it is a semisimple Artinian ring with just one irreducible module, namely, $D$ itself. So every $D$-module is isomorphic to a direct sum of copies of $D$ (e.g. every vector space has a basis!). Let $P = D^{\oplus n}$, a projective generator. Then, $\text{End}_D(P)^{\text{op}} \cong M_n(D)$ is Morita equivalent to $D$. The equivalence of categories between $D$ and $M_n(D)$ is given explicitly in one direction by tensoring over $D$ with the $M_n(D)$-$D$-bimodule of column vectors, and in the other direction by tensoring over $M_n(D)$ with the $D,M_n(D)$-bimodule of row vectors. Thus every $M_n(D)$-module is isomorphic to a direct sum of copies of the $n$-dimensional module of column vectors. I stress this example because by Wedderburn’s theorem, all simple Artinian rings are isomorphic to $M_n(D)$ for some division ring $D$.

**Example 1.2. Symmetric algebras.** I want to discuss the case of a symmetric algebra over a finite dimensional vector space. It is customary to work with the dual space... So, let $V \neq 0$ be a finite dimensional vector space over an algebraically closed field $k$ (to make life easy). Let $x_1, \ldots, x_n$ be a basis for $V^*$. Then, the symmetric algebra $S(V^*)$ can be identified with the polynomial ring $k[x_1, \ldots, x_n]$, and we can think of its elements as functions on $V$. Hilbert’s basis theorem says that $S(V^*)$ is Noetherian. It is not Artinian!!! The Nullstellensatz shows that the irreducible $S(V^*)$-modules are in 1–1 correspondence with the points in the vector space $V$, $v \in V$ corresponding to the one dimensional irreducible module $k_v$ on which $f \in S(V^*)$ acts as multiplication by the scalar $f(v)$. The annihilator of the module $k_v$ is the maximal ideal $I_v$ of $S(V^*)$ consisting of all functions that are zero on the point $v$. The Jacobson radical $J(S(V))$ is the intersection of the annihilators of all the points $v \in V$, hence it is the set of all functions that are zero on all of $V$. By the Nullstellensatz again, that is zero. Hence, $J(S(V^*)) = 0$.

**Example 1.3. Polynomials in one variable.** You should also recall the special case that $A = k[x]$ is a polynomial ring in one variable over an algebraically closed field. In that case, $A$ is a PID so we have an especially good theory for finitely generated modules. Indeed, if $M$ is a finitely generated $A$-module, then it splits uniquely as a direct sum of a torsion part and a free part. So suppose that $M$ is torsion. Then it is a finite dimensional vector space and the $A$-module structure is completely determined by the endomorphism defined by the action of $x$. Now you can put this endomorphism into Jordan normal form, and deduce that the indecomposable summands of $M$ are precisely the Jordan blocks. Thus you get a complete classification of the finitely generated indecomposable modules: either $k[x]$ itself, or the $n \times n$ Jordan block $J_n(\lambda)$ of eigenvalue $\lambda \in k$. The irreducible modules are the $J_n(\lambda)$’s. Note the module $J_n(\lambda)$ is a uniserial module, meaning it has...
a unique composition series, and all the composition factors are isomorphic to \( J_1(\lambda) \).

**Example 1.4. Exterior algebras.** So much for commutative algebra. What about skew-commutative algebra? Let \( V \) be a finite dimensional vector space of dimension \( n \). Consider the exterior algebra \( A = \bigwedge V = \bigoplus_{d \geq 0} \bigwedge^d V \) of dimension \( 2^n \). Since it is finite dimensional, it is Artinian. The set of all non-units in \( \bigwedge V \) is precisely the left ideal \( \bigoplus_{d>0} \bigwedge^d V \). Hence this must be the unique maximal left ideal, so is the Jacobson radical, and the quotient is the field \( k \). Since the Jacobson radical acts as zero on any completely reducible module, the irreducible modules of \( A \) are precisely the irreducible modules of \( A/J(A) \). But that is the field \( k \). Therefore there is a unique irreducible module, namely the field \( k \) itself.

**Example 1.5. Group algebras.** Let \( G \) be a finite group. Then the group algebra \( kG \), \( k \) a field, is the algebra equal to the vector space with basis the elements of \( G \) and with multiplication given by extending the multiplication in the group \( G \) by bilinearity. Since \( kG \) is a finite dimensional algebra, it is Artinian. You probably remember Maschke’s theorem: the algebra \( kG \) is semisimple if and only if \( \text{char } k \nmid |G| \). Since it is so important, let’s run through the proof.

Suppose that \( \text{char } k \nmid |G| \). Let \( M \) be a \( kG \)-module and let \( N \) be a submodule. Let \( \pi : M \twoheadrightarrow N \) be any linear map extending the identity map on \( N \). For \( g \in G \), consider \( g^{-1} \circ \pi \circ g : M \to N \). It also extends the identity map on \( N \). Hence so does

\[
\frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \pi \circ g.
\]

But that is now even \( G \)-equivariant. Hence its kernel is a \( G \)-stable complement to \( N \) in \( M \). We’ve shown every submodule of a \( G \)-module has a complement, which means \( kG \) is a semisimple algebra. Conversely, suppose that \( \text{char } k = p ||G| \). Let \( e = \sum_{g \in G} g \). Since \( ge = e = eg \) for each \( g \in G \), \( e \) spans a one dimensional ideal in \( kG \). Since \( e^2 = 0 \), this ideal is nilpotent, so it is contained in the Jacobson radical because in an Artinian ring, \( J(R) \) is the sum of all the nilpotent ideals of \( R \). Hence, \( J(kG) \neq 0 \) and \( kG \) is not semisimple.

By the way, when talking about \( kG \)-modules, people often use an alternative language and call a \( kG \)-module \( M \) instead a representation of \( G \). That is because the action of \( G \) on the module \( M \) induces a group homomorphism \( \rho : G \to GL(M) \) which “represents” the group as a group of invertible \( \dim M \times \dim M \) matrices.

Assume from now on that \( k \) is algebraically closed of characteristic 0. By Schur’s lemma (in its strong form for finite dimensional modules and an algebraically closed field) the endomorphism algebra of a simple module is just \( k \). So by Wedderburn’s theorem, \( kG = M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k) \), where the number \( r \) is the number of inequivalent irreducible representations, and
$n_1, \ldots, n_r$ are the dimensions of the respective simple modules. Question: what is $r$ exactly? Well, consider the center $Z(kG)$. It is $r$-dimensional, since you have the scalar matrices in each $M_{n_i}(k)$. On the other hand, an easy calculation shows that if $\sum_{g \in G} c_g g \in kG$ is a central element, then the coefficients $c_g$ must be constant on each conjugacy class of $G$. Hence, the dimension of the center $Z(kG)$ is equal to the number of conjugacy classes in $G$. Therefore: the number of inequivalent irreducible $kG$-modules is equal to the number of conjugacy classes in $G$. Moreover, their dimensions $n_1, \ldots, n_r$ satisfy

$$|G| = n_1^2 + \cdots + n_r^2.$$ 

This is the starting point for character theory of finite groups, which provides many more wonderful numerical connections between the structure of the group and its representations.

**Example 1.6. Abelian and cyclic groups.** Let $G$ be a finite abelian group and $k$ be an algebraically closed field of characteristic 0. We’ve just seen that there are $|G|$ inequivalent irreducible representations, and they must all be one dimensional.

Take for instance $G = C_n$, a cyclic group. Then we can easily construct all the one dimensional irreducibles, as follows. Let $g \in G$ be a generator. Let $\omega$ be a primitive $n$th root of unity in $k$. Then the $r$th irreducible representation is the field $k$ on which $g$ acts as the scalar $\omega^r$, for $r = 0, 1, \ldots, n - 1$.

There’s another way to see this: the group algebra $kC_n$ is the quotient of the polynomial algebra $k[x]$ by the ideal $(x^n - 1)$. Since $(x^n - 1) = (x - 1)(x - \omega)(x - \omega^2)\ldots(x - \omega^{n-1})$ and these are relatively prime factors, the Chinese Remainder Theorem shows that

$$kC_n \cong k[x]/(x - 1) \oplus \cdots \oplus k[x]/(x - \omega^{n-1}).$$

Hence we’ve decomposed the group algebra as a direct sum of $1 \times 1$ matrix algebras! This approach lets you get a glimpse of what happens when the field is not algebraically closed: it is all about how $(x^n - 1)$ can be factorized over your ground field.

**Exercise 4.** Classify the indecomposable modules of the group algebra $kC_n$ of the cyclic group of order $n$ over an algebraically closed field of characteristic $p$. 