

## 1. REVIEW OF SOME BASIC THEORY

In this chapter, I want to quickly run through some basic theorems which you probably remember either from 600 algebra or from Frank's course. A good reference for a slightly more detailed review if you want it is chapter one of Benson's book "Representations and cohomology I".

- (1) *The Jordan-Hölder theorem.* A composition series for an  $R$ -module  $M$  is a (finite) series of submodules

$$0 = M_0 < M_1 < \cdots < M_n = M$$

such that each  $M_i/M_{i-1}$  is *irreducible* (or *simple*). The Jordan-Hölder theorem says that any two (finite) series of submodules of an arbitrary module  $M$  can always be refined to series of equal length such that the factors in one series are isomorphic to the factors in the other series, possibly up to a permutation.

- (2) *Chain conditions.* A module  $M$  satisfies DCC if every descending chain of submodules eventually stops, and ACC if every ascending chain eventually stops. By the Jordan-Hölder theorem,  $M$  has a composition series if and only if it satisfies both ACC and DCC, in which case any series of submodules of  $M$  can be refined to a composition series. In that case, for a simple module  $L$ , the *composition multiplicity*  $[M : L]$  counting the number of factors of a composition series of  $M$  that are isomorphic to  $L$  is a well-defined invariant of  $M$ .
- (3) *Noetherian rings.* A ring  $R$  is called *Noetherian* if the *regular module*  ${}_R R$  satisfies ACC on submodules (a.k.a. left ideals). All the rings we'll meet in this course will be Noetherian. If  $R$  is Noetherian, every finitely generated  $R$ -module satisfies ACC. Moreover, every submodule of a finitely generated  $R$ -module is also finitely generated.
- (4) *Completely reducible modules.* The *socle* of an  $R$ -module  $M$  is the sum of all the irreducible submodules of  $M$ , written  $\text{soc } M$ . A module  $M$  is called *completely reducible* (or *semisimple*) if  $M = \text{soc } M$ , which by Zorn's lemma is equivalent to the statement that every submodule of  $M$  has a complement, or to the statement that  $M$  is a direct sum of irreducible modules. By its definition, the socle  $\text{soc } M$  is the largest completely reducible submodule of  $M$ . The *radical* of  $M$  is the intersection of all the maximal submodules of  $M$ , written  $\text{rad } M$ . If  $M$  satisfies DCC, then  $M$  is semisimple if and only if  $\text{rad } M = 0$ . So assuming that  $M$  satisfies DCC,  $M/\text{rad } M$  is the largest completely reducible quotient of  $M$ .
- (5) *The Jacobson radical.* The Jacobson radical  $J(R)$  of  $R$  is the intersection of the maximal left ideals of  $R$ , or equivalently, the intersection of the annihilators of all the simple  $R$ -modules. The latter description makes it clear that  $J(R)$  is a two-sided ideal of  $R$ . Note

tautologically that  $J(R) = \text{rad}({}_R R)$ . We have *Nakayama's lemma*: If  $M$  is a finitely generated  $R$ -module and  $J(R)M = M$  then  $M = 0$ .

- (6) *Artinian rings*. A ring is called *Artinian* if  $R$  satisfies DCC on left ideals. In that case, (i) every finitely generated  $R$ -module has DCC on submodules; (ii) by Nakayama's lemma,  $\text{rad}(M) = J(R)M$  for any finitely generated  $M$ ; (iii)  $J(R)$  is nilpotent, i.e.  $J(R)^n = 0$  for some  $n$ . Now suppose that  $M$  is a finitely generated  $R$ -module, and let  $M_i = J(R)^i M$ . Then,  $\text{rad}(M_i/M_{i+1}) = J(R)(M_i/M_{i+1}) = 0$ , so  $M_i/M_{i+1}$  is completely reducible. Since  $M$  has DCC, so does  $M_i/M_{i+1}$ , hence it is a finite direct sum of irreducible modules, hence it has a composition series. Since  $J(R)$  is nilpotent,  $M_n = 0$  for some  $n$ , hence  $M$  itself has a composition series. This shows: if  $R$  is Artinian, then every finitely generated  $R$ -module has a composition series. Note applying this to the regular module, you get that Artinian rings are Noetherian.
- (7) *Schur's lemma*. Schur's lemma says that if  $M \not\cong N$  are irreducible  $R$ -modules, then  $\text{Hom}_R(M, N) = 0$ , while  $\text{End}_R(M, M)$  is a division ring. Thus, if  $M$  is a finite direct sum of irreducible  $R$ -modules, say  $M = M_1^{\oplus n_1} \oplus \cdots \oplus M_r^{\oplus n_r}$  with  $M_i \not\cong M_j$ , you get that  $\text{End}_R(M) \cong M_{n_1}(\text{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\text{End}_R(M_r))$ , a direct sum of matrix algebras over division rings. There is a stronger form of Schur's lemma too when  $R$  is an algebra over an algebraically closed field  $k$  and  $M$  is a finite dimensional irreducible  $R$ -module: in that case,  $\text{End}_R(M) = k$ . Proof: take  $f \in \text{End}_R(M)$ . Let  $\lambda \in k$  be an eigenvalue. Then,  $\ker(f - \lambda)$  is a non-zero  $R$ -submodule of  $M$ , so it's all of  $M$  as  $M$  is irreducible. Hence,  $f = \lambda$  is a scalar and  $\text{End}_R(M) = k$ .
- (8) *Wedderburn's theorem*. A ring  $R$  is called *semisimple* if  $J(R) = 0$  (so  $R/J(R)$  is *always* a semisimple ring!). Assuming that  $R$  is Artinian,  $R$  is semisimple if and only if every  $R$ -module is completely reducible, which is if and only if the regular module is completely reducible. So we can decompose  ${}_R R = M_1^{\oplus n_1} \oplus \cdots \oplus M_r^{\oplus n_r}$  for irreducible modules  $M_i \not\cong M_j$ , and get by Schur's lemma that

$$R \cong \text{End}_R({}_R R)^{\text{op}} \cong M_{n_1}(\text{End}_R(M_1)) \oplus \cdots \oplus M_{n_r}(\text{End}_R(M_r)),$$

a direct sum of matrix algebras over division rings. Moreover,  $R$  has exactly  $r$  different irreducible modules up to isomorphism, of dimensions  $n_1, \dots, n_r$ , namely the modules of column vectors over each of the matrix algebras. Hence, the numbers  $r, n_1, \dots, n_r$  and the division rings  $\text{End}_R(M_i)$  are uniquely determined by  $R$ . Conversely, any finite direct sum of matrix algebras over division rings is a semisimple Artinian ring.

- (9) *Fitting's lemma*. Suppose the  $R$ -module  $M$  has a composition series (i.e. both ACC and DCC). Let  $f \in \text{End}_R(M)$ . Fitting's lemma says that for sufficiently large  $n$ ,  $M = \text{im}(f^n) \oplus \ker(f^n)$ . Let me

explain the main application of Fitting's lemma. An  $R$ -module is called *indecomposable* if it cannot be written as a direct sum of two non-zero submodules. A *local ring* is a ring  $R$  with a unique maximal left ideal (which must therefore be equal to its Jacobson radical since that is the intersection of all the maximal left ideals). Now I claim that *if  $M$  is an indecomposable module having a composition series, then  $\text{End}_R(M)$  is a local ring.* Let  $I$  be a maximal left ideal of  $E = \text{End}_R(M)$ . Pick  $a \notin I$ . We need to show that  $Ea = E$ . Well,  $E = Ea + I$ , so we can write  $1_E = \lambda a + \mu$  with  $\lambda \in E, \mu \in I$ . By Fitting's lemma  $M = \text{im}(\mu^n) \oplus \ker(\mu^n)$  for some  $n$ . But  $M$  is indecomposable, so either  $\mu^n$  is onto or  $\mu^n = 0$ . The former cannot occur since  $\mu$  is not a unit, so it is not an automorphism of  $M$ . Hence,  $\mu^n = 0$ . Now  $(1 + \mu + \cdots + \mu^{n-1})\lambda a = (1 + \cdots + \mu^{n-1})(1 - \mu) = 1 - \mu^n = 1$ , so  $Ea = E$ . Using this you can now easily show that if  $R$  is an Artinian ring, then a finitely generated  $R$ -module  $M$  is indecomposable if and only if  $\text{End}_R(M)$  is a local ring.

- (10) *The Krull-Schmidt theorem.* Suppose  $R$  is Artinian. Then, every finitely generated  $R$ -module decomposes uniquely up to isomorphism as a direct sum of indecomposable modules. You can show more generally that if  $R$  is any old ring and  $M$  is an  $R$ -module that is a direct sum of finitely many indecomposables  $M_i$  such that each  $\text{End}_R(M_i)$  is a local ring, then all other decompositions of  $M$  as a direct sum of indecomposables are isomorphic to the given one.
- (11) *Projective modules.* Remember that a module  $P$  is called *projective* if every map from  $P$  to a quotient of a module  $M$  lifts to a map from  $P$  to  $M$  itself. This is equivalent to the statement that every map  $M \twoheadrightarrow P$  splits, and that  $P$  is a summand of a free module. A *projective generator* means a finitely generated projective  $R$ -module  $P$  such that every  $R$ -module is a quotient of a direct sum of copies of  $P$ .
- (12) *The Morita theorem.* Let  $R$  and  $S$  be rings. The following are equivalent: (i)  $R\text{-Mod}$  and  $S\text{-Mod}$  are equivalent categories; (ii)  $R\text{-mod}$  and  $S\text{-mod}$  are equivalent categories; (iii) there exist an  $R, S$ -bimodule  ${}_R M_S$  and an  $S, R$ -bimodule  ${}_S N_R$  such that  $M \otimes_S N \cong R$  as an  $R, R$ -bimodule and  $N \otimes_R M \cong S$  as an  $S, S$ -bimodule; (iv) there is a projective generator  $P$  for  $R$  such that  $S \cong \text{End}_R(P)^{\text{op}}$ . In that case,  $R$  and  $S$  are said to be *Morita equivalent*.

**Exercise 2.** (i) Prove that  $\text{rad}(M \oplus N) = \text{rad } M \oplus \text{rad } N$ .

(ii) Prove directly that the algebra  $M_n(D)$  of  $n \times n$  matrices over a division ring is a simple ring, i.e. it has no non-trivial two-sided ideals.

(iii) Deduce from (i) and (ii) that a finite direct sum of matrix algebras over division rings is a semisimple ring.

**Exercise 3.** Let  $M$  be an  $R$ -module, and let  $X, Y$  be submodules such that  $M/X$  is semisimple and  $M/Y$  is irreducible. Prove that  $M/(X \cap Y)$  is

semisimple. Hence prove the statement made in (4) above that if  $M$  satisfies DCC, then  $M$  is semisimple if and only if  $\text{rad } M = 0$ .

Let me now give a few more basic examples.

**Example 1.1.** *Division rings.* Let  $D$  be a division ring (e.g. a field!). Since  $D$  is a simple  $D$ -module, it is a semisimple Artinian ring with just one irreducible module, namely,  $D$  itself. So every  $D$ -module is isomorphic to a direct sum of copies of  $D$  (e.g. every vector space has a basis!). Let  $P = D^{\oplus n}$ , a projective generator. Then,  $\text{End}_D(P)^{\text{op}} \cong M_n(D)$  is Morita equivalent to  $D$ . The equivalence of categories between  $D$  and  $M_n(D)$  is given explicitly in one direction by tensoring over  $D$  with the  $M_n(D), D$ -bimodule of column vectors, and in the other direction by tensoring over  $M_n(D)$  with the  $D, M_n(D)$ -bimodule of row vectors. Thus every  $M_n(D)$ -module is isomorphic to a direct sum of copies of the  $n$ -dimensional module of column vectors. I stress this example because by Wedderburn's theorem, all simple Artinian rings are isomorphic to  $M_n(D)$  for some division ring  $D$ .

**Example 1.2.** *Symmetric algebras.* I want to discuss the case of a symmetric algebra over a finite dimensional vector space. It is customary to work with the dual space... So, let  $V \neq 0$  be a finite dimensional vector space over an algebraically closed field  $k$  (to make life easy). Let  $x_1, \dots, x_n$  be a basis for  $V^*$ . Then, the symmetric algebra  $S(V^*)$  can be identified with the polynomial ring  $k[x_1, \dots, x_n]$ , and we can think of its elements as functions on  $V$ . Hilbert's basis theorem says that  $S(V^*)$  is Noetherian. It is *not* Artinian!!! The Nullstellensatz shows that the irreducible  $S(V^*)$ -modules are in 1-1 correspondence with the points in the vector space  $V$ ,  $v \in V$  corresponding to the one dimensional irreducible module  $k_v$  on which  $f \in S(V^*)$  acts as multiplication by the scalar  $f(v)$ . The annihilator of the module  $k_v$  is the maximal ideal  $I_v$  of  $S(V^*)$  consisting of all functions that are zero on the point  $v$ . The Jacobson radical  $J(S(V^*))$  is the intersection of the annihilators of all the points  $v \in V$ , hence it is the set of all functions that are zero on all of  $V$ . By the Nullstellensatz again, that is zero. Hence,  $J(S(V^*)) = 0$ .

**Example 1.3.** *Polynomials in one variable.* You should also recall the special case that  $A = k[x]$  is a polynomial ring in one variable over an algebraically closed field. In that case,  $A$  is a PID so we have an especially good theory for finitely generated modules. Indeed, if  $M$  is a finitely generated  $A$ -module, then it splits uniquely as a direct sum of a torsion part and a free part. So suppose that  $M$  is torsion. Then it is a finite dimensional vector space and the  $A$ -module structure is completely determined by the endomorphism defined by the action of  $x$ . Now you can put this endomorphism into Jordan normal form, and deduce that the indecomposable summands of  $M$  are precisely the Jordan blocks. Thus you get a complete classification of the finitely generated indecomposable modules: either  $k[x]$  itself, or the  $n \times n$  Jordan block  $J_n(\lambda)$  of eigenvalue  $\lambda \in k$ . The irreducible modules are the  $J_1(\lambda)$ 's. Note the module  $J_n(\lambda)$  is a *uniserial module*, meaning it has

a unique composition series, and all the composition factors are isomorphic to  $J_1(\lambda)$ .

**Example 1.4. Exterior algebras.** So much for commutative algebra. What about skew-commutative algebra? Let  $V$  be a finite dimensional vector space of dimension  $n$ . Consider the exterior algebra  $A = \bigwedge V = \bigoplus_{d \geq 0} \bigwedge^d V$  of dimension  $2^n$ . Since it is finite dimensional, it is Artinian. The set of all non-units in  $\bigwedge V$  is precisely the left ideal  $\bigoplus_{d > 0} \bigwedge^d V$ . Hence this must be the unique maximal left ideal, so is the Jacobson radical, and the quotient is the field  $k$ . Since the Jacobson radical acts as zero on any completely reducible module, the irreducible modules of  $A$  are precisely the irreducible modules of  $A/J(A)$ . But that is the field  $k$ . Therefore there is a unique irreducible module, namely the field  $k$  itself.

**Example 1.5. Group algebras.** Let  $G$  be a finite group. Then the group algebra  $kG$ ,  $k$  a field, is the algebra equal to the vector space with basis the elements of  $G$  and with multiplication given by extending the multiplication in the group  $G$  by bilinearity. Since  $kG$  is a finite dimensional algebra, it is Artinian. You probably remember *Maschke's theorem*: the algebra  $kG$  is semisimple if and only if  $\text{char } k \nmid |G|$ . Since it is so important, let's run through the proof.

Suppose that  $\text{char } k \nmid |G|$ . Let  $M$  be a  $kG$ -module and let  $N$  be a submodule. Let  $\pi : M \rightarrow N$  be any linear map extending the identity map on  $N$ . For  $g \in G$ , consider  $g^{-1} \circ \pi \circ g : M \rightarrow N$ . It also extends the identity map on  $N$ . Hence so does

$$\frac{1}{|G|} \sum_{g \in G} g^{-1} \circ \pi \circ g.$$

But that is now even  $G$ -equivariant. Hence its kernel is a  $G$ -stable complement to  $N$  in  $M$ . We've shown every submodule of a  $G$ -module has a complement, which means  $kG$  is a semisimple algebra. Conversely, suppose that  $\text{char } k = p \mid |G|$ . Let  $e = \sum_{g \in G} g$ . Since  $ge = e = eg$  for each  $g \in G$ ,  $e$  spans a one dimensional ideal in  $kG$ . Since  $e^2 = 0$ , this ideal is nilpotent, so it is contained in the Jacobson radical because in an Artinian ring,  $J(R)$  is the sum of all the nilpotent ideals of  $R$ . Hence,  $J(kG) \neq 0$  and  $kG$  is not semisimple.

By the way, when talking about  $kG$ -modules, people often use an alternative language and call a  $kG$ -module  $M$  instead a *representation of  $G$* . That is because the action of  $G$  on the module  $M$  induces a group homomorphism  $\rho : G \rightarrow GL(M)$  which "represents" the group as a group of invertible  $\dim M \times \dim M$  matrices.

Assume from now on that  $k$  is algebraically closed of characteristic 0. By Schur's lemma (in its strong form for finite dimensional modules and an algebraically closed field) the endomorphism algebra of a simple module is just  $k$ . So by Wedderburn's theorem,  $kG = M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$ , where the number  $r$  is the number of inequivalent irreducible representations, and

$n_1, \dots, n_r$  are the dimensions of the respective simple modules. Question: what is  $r$  exactly? Well, consider the center  $Z(kG)$ . It is  $r$ -dimensional, since you have the scalar matrices in each  $M_{n_i}(k)$ . On the other hand, an easy calculation shows that if  $\sum_{g \in G} c_g g \in kG$  is a central element, then the coefficients  $c_g$  must be constant on each conjugacy class of  $G$ . Hence, the dimension of the center  $Z(kG)$  is equal to the number of conjugacy classes in  $G$ . Therefore: *the number of inequivalent irreducible  $kG$ -modules is equal to the number of conjugacy classes in  $G$* . Moreover, their dimensions  $n_1, \dots, n_r$  satisfy

$$|G| = n_1^2 + \dots + n_r^2.$$

This is the starting point for character theory of finite groups, which provides many more wonderful numerical connections between the structure of the group and its representations.

**Example 1.6.** *Abelian and cyclic groups.* Let  $G$  be a finite abelian group and  $k$  be an algebraically closed field of characteristic 0. We've just seen that there are  $|G|$  inequivalent irreducible representations, and they must all be one dimensional.

Take for instance  $G = C_n$ , a cyclic group. Then we can easily construct all the one dimensional irreducibles, as follows. Let  $g \in G$  be a generator. Let  $\omega$  be a primitive  $n$ th root of unity in  $k$ . Then the  $r$ th irreducible representation is the field  $k$  on which  $g$  acts as the scalar  $\omega^r$ , for  $r = 0, 1, \dots, n-1$ .

There's another way to see this: the group algebra  $kC_n$  is the quotient of the polynomial algebra  $k[x]$  by the ideal  $(x^n - 1)$ . Since  $(x^n - 1) = (x - 1)(x - \omega) \dots (x - \omega^{n-1})$  and these are relatively prime factors, the Chinese Remainder Theorem shows that

$$kC_n \cong k[x]/(x - 1) \oplus \dots \oplus k[x]/(x - \omega^{n-1}).$$

Hence we've decomposed the group algebra as a direct sum of  $1 \times 1$  matrix algebras! This approach lets you get a glimpse of what happens when the field is not algebraically closed: it is all about how  $(x^n - 1)$  can be factorized over your ground field.

**Exercise 4.** Classify the indecomposable modules of the group algebra  $kC_n$  of the cyclic group of order  $n$  over an algebraically closed field of characteristic  $p$ .