Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra with simple roots $\{\alpha_i\}_{i \in I}$, Chevalley generators $\{e_i, f_i\}_{i \in I}$, and weight lattice $X$. Let $\mathcal{C} = \bigoplus_{\lambda \in X} C_\lambda$ be an abelian category with each $C_\lambda$ equivalent to modules over some finite dimensional algebra over a ground field $\mathbb{k}$.

A categorical action of $\mathfrak{g}$ on $\mathcal{C}$ means:

- Biadjoint functors $E_i : C_\lambda \to C_{\lambda + \alpha_i}$, $F_i : C_{\lambda + \alpha_i} \to C_\lambda$ making the Grothendieck group $[\mathcal{C}] := \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathcal{C})$ into an integrable $\mathfrak{g}$-module with (finite dimensional) weight spaces $[C_\lambda]$.

- Natural transformations $\xi = \uparrow \in \text{End}(F_i)$, $\tau = \leftrightarrow \in \text{Hom}(F_i F_j, F_j F_i)$ inducing a locally nilpotent action of the quiver Hecke algebra $QH_d$ of type $\mathfrak{g}$ on $F^d$ for each $d \geq 1$. Here $F := \bigoplus_{i \in I} F_i$; also $E := \bigoplus_{i \in I} E_i$.

By a theorem of Rouquier, this is equivalent to $\mathcal{C}$ being a 2-representation of Rouquier’s Kac-Moody 2-category $\hat{\mathcal{U}}(\mathfrak{g})$.

Khovanov and Lauda defined a related 2-category $\mathcal{U}(\mathfrak{g})$ which satisfies some additional cyclicity and the “infinite Grassmannian” relation; by a strong categorical action we mean a 2-representation of $\mathcal{U}(\mathfrak{g})$. 
Degenerate affine Hecke algebras

There is a variation on the definition of categorical action in which the degenerate affine Hecke algebra $AH_d$ replaces $QH_d$. Assume for this that $\mathfrak{g} = \mathfrak{sl}_\infty$ if $p = 0$ or $\hat{\mathfrak{sl}}_p$ if $p > 0$ where $p := \text{char } k$. Identify $I \leftrightarrow \mathbb{Z} \cdot 1_k \subseteq k$ so $\alpha_i$ and $\alpha_j$ are adjacent iff $i = j \pm 1$ in $k$. Then the second part of the definition of categorical action becomes:

- Natural transformations $x = \uparrow \in \text{End}(F)$, $t = \uparrow \uparrow \in \text{End}(F^2)$ such that each $F_i$ is the generalized $i$-eigenspace of $x$ on $F$ and the local dAHA relations hold.

Local dAHA relations

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
\downarrow & \downarrow & \downarrow \\
\end{array} & = & \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \\
\begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} & = & \begin{array}{c}
\uparrow \\
\downarrow \\
\end{array} \\
\end{array} \\
t^2 & = 1, \\
t_2 t_1 t_2 & = t_1 t_2 t_1, \\
x_2 t & = tx_1 + 1.
\end{align*}
\]

The equivalence with the earlier definition depends on an isomorphism $\hat{QH}_d \cong \hat{AH}_d$ for suitable completions (Rouquier, B.–Kleshchev).
Example: categorification of highest weight modules

Let $\lambda = \sum_{i \in I} a_i \varpi_i \in X^+$ be a dominant weight. Let $V(\lambda)$ denote the corresponding integrable highest weight module for $\mathfrak{g}$.

**Theorem (Kang-Kashiwara)**

Let $QH^\lambda_d := QH_d / \langle \xi_i^{1_i} \mid i = (i_1, \ldots, i_d) \in I^d \rangle$ be the cyclotomic $QHA$. Then the category $\mathcal{V}(\lambda) := \bigoplus_{d \geq 0} \text{Rep}(QH^\lambda_d)$ admits a categorical $\mathfrak{g}$-action with the functors $E$ and $F$ arising from restriction and induction between $QH^\lambda_d$ and $QH^\lambda_{d+1}$ (for all $d$). Moreover the complexified Grothendieck group $[\mathcal{V}(\lambda)]$ is canonically isomorphic to $V(\lambda)$.

Webster has a different proof which establishes further that $\mathcal{V}(\lambda)$ admits a strong categorical action.

In symmetric case, $\text{char } k = 0$, results of Varagnolo-Vasserot imply that the classes of the PIMs correspond to the canonical basis of $V(\lambda)$.

If $\mathfrak{g} = \mathfrak{sl}_\infty$ or $\hat{\mathfrak{sl}}_p$ according to $k$, the category $\mathcal{V}(\lambda)$ is equivalent to $\bigoplus_{d \geq 0} \text{Rep}(AH^\lambda_d)$ where $AH^\lambda_d := AH_d / \langle \prod_{i \in I} (x_1 - i)^{a_i} \rangle$ is the cyclotomic $dAHA$. This is (part of) Ariki’s categorification theorem (known before).
Example: category $O$ is a categorification of tensor space

Assume $k = \mathbb{C}$ and $g = sl_\infty$, so $l = \mathbb{Z}$ as above. Let $V$ be the natural $g$-module on basis $\{v_i \mid i \in \mathbb{Z}\}$, so $E_i v_j = \delta_{i+1,j} v_i$, $F_i v_j = \delta_{i,j} v_{i+1}$.

Theorem

Let $C$ be the integral weight part of the Bernstein-Gelfand-Gelfand category $O$ for the general linear Lie algebra $\mathfrak{gl}_n(k)$. Then $C$ admits a categorical $g$-action with:

- $F := k^n \otimes ?$, i.e. it is tensoring with the module of column vectors;
- $x \in \text{End}(F)$ defined by the Casimir tensor $\sum_{i,j=1}^n e_{i,j} \otimes e_{j,i}$;
- $t \in \text{End}(F^2)$ induced by the flip $k^n \otimes k^n \rightarrow k^n \otimes k^n$, $u \otimes v \mapsto v \otimes u$.

This example is a highest weight category. Its Grothendieck group identifies with the $g$-module $V \otimes^n$ so that the classes of the Verma modules $\Delta(\lambda)$ correspond to the monomials $v_{(\lambda+\rho,\epsilon_1)} \otimes \cdots \otimes v_{(\lambda+\rho,\epsilon_n)} \in V \otimes^n$.

The Kazhdan-Lusztig conjecture for $\mathfrak{gl}_n(k)$ implies that the classes of the PIMs correspond to Lusztig’s canonical basis for $V \otimes^n$. 
Uniqueness of minimal categorifications

Say $\mathcal{C}$ is a minimal categorification of $V(\lambda)$ if $[\mathcal{C}] \cong V(\lambda)$ and its highest weight subcategory $\mathcal{C}_\lambda$ is a copy of $\text{Vec}$. Example: $\mathcal{V}(\lambda)$.

**Theorem (Rouquier)**

All minimal categorifications of $V(\lambda)$ are strongly equivariantly equivalent.

This is the first part of the Jordan-Hölder theorem for categorical actions, which Chuang and Rouquier used to construct of an action of the braid group associated to $\mathfrak{g}$ on the bounded derived category of an arbitrary categorical action. When applied to the minimal categorification $\bigoplus_{d \geq 0} \text{Rep}(kS_d)$ of $V(\omega_0)$ for $\mathfrak{g} = \hat{\mathfrak{sl}}_p$, this was the key ingredient in their proof of Broué’s abelian defect conjecture for the symmetric groups. Further structural results for strong categorical actions are also beginning to emerge, e.g. for $\mathfrak{g} = \mathfrak{sl}_2$ Beliakova-Habiro-Lauda-Zivkovic have recently shown that $\mathfrak{g}[t]$ acts on the center of any strong categorification.
Definition of tensor product categorification

Suppose we are given $\lambda_1, \ldots, \lambda_r \in X^+$. Let

- $\Lambda := B(\lambda_1) \times \cdots \times B(\lambda_r)$—product of highest weight crystals;
- $\Xi := P(\lambda_1) \times \cdots \times P(\lambda_r)$—product of the underlying sets of weights.

Note $\Lambda \rightarrow \Xi$. The reverse dominance ordering on $\Xi$ is defined from:

$$(\mu_1, \ldots, \mu_r) \leq (\nu_1, \ldots, \nu_r) \iff \mu_1 + \cdots + \mu_s \geq \nu_1 + \cdots + \nu_s \quad \forall s.$$ 

A tensor product categorification of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ is a category $C$ equipped with a categorical $g$-action such that:

- $C$ is standardly stratified with standard objects $\{\Delta(\pi) \mid \pi \in \Lambda\}$ and poset $\Xi$. Let $C^0 = \bigoplus_{\xi \in \Xi} C_{\leq \xi}/C_{< \xi}$ be the associated graded category and $\Delta : C^0 \rightarrow C$ be the (exact) standardization functor.
- $C^0$ has additional structure of a minimal categorification of the $g^\oplus r$-module $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$, consistent with labelling of its simples.
- $F_i \circ \Delta$ is filtered by $\Delta \circ 1 F_i, \ldots, \Delta \circ r F_i$, and similarly for $E_i$.

These axioms imply that $[C] \cong [C^\Delta] \cong [C^0] \cong V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ with $F_i$ acting as $\sum_{s=1}^r 1^\otimes (s-1) \otimes F_i \otimes 1^\otimes (r-s)$, etc...
The double centralizer property

Let $C$ be a tensor product categorification of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$. Set $\lambda := \lambda_1 + \cdots + \lambda_r$. Recall $\Lambda = B(\lambda_1) \times \cdots \times B(\lambda_r)$ labels the simples $\{L(\pi)\}$ in $C$. Kashiwara’s tensor product rule defines a crystal structure on $\Lambda$. Let $\Lambda^\circ$ be the copy of $B(\lambda)$ generated by the highest weight vector. Let $\overline{C}$ be the quotient of $C$ by the Serre subcategory generated by $\{L(\pi) \mid \pi \in \Lambda \setminus \Lambda^\circ\}$. This is a minimal categorification of $V(\lambda)$.

**Theorem (Losev-Webster)**

The quotient functor $\pi : C \to \overline{C}$ is fully faithful on projectives. Hence all tensor product categorifications of $V(\lambda_1) \otimes \cdots \otimes V(\lambda_r)$ are strongly equivariantly equivalent.

This establishes uniqueness of TPCs.
Webster earlier proved existence using certain “tensor product algebras,” which actually admit strong categorical actions. Hence any TPC is strong.

In finite type $A$ there is another construction involving parabolic category $O$. It is related to Schur-Weyl duality for higher levels (B.-Kleshchev).
Some minuscule tensor product categorifications

Assume henceforth that $k = \mathbb{C}$ and $\mathfrak{g} = \mathfrak{sl}_\infty$. Let $V^0$ be the natural $\mathfrak{g}$-module, let $V^1$ be the dual natural module. Consider the problem of categorifying some tensor product $M := \bigwedge^{n_1} V^{c_1} \otimes \cdots \otimes \bigwedge^{n_r} V^{c_r}$ for some $n_1, \ldots, n_r \geq 1, c_1, \ldots, c_r \in \{0, 1\}$. It is integrable but with infinite dimensional weight spaces. Also the reverse dominance ordering has infinite chains. Our earlier definition of TPC needs a little massaging!

Theorem (B.-Losev-Webster)

Tensor product categorifications of $M$ exist and are unique. Any such is a highest weight category, it has a unique graded lift so that the 2-morphisms in $\hat{U}(\mathfrak{g})$ act by homogeneous natural transformations, and this graded lift is standard Koszul.

The proof involves some careful direct limit argument starting from Losev-Webster’s uniqueness theorem for finite $\mathfrak{sl}_n$. 

Jonathan Brundan (University of Oregon)  Schur-Weyl duality and categorification  18 August 2014 9 / 11
Example: category $\mathcal{O}$ for the general linear superalgebra

Let $M$ be as on the previous slide. Here is the construction of a TPC of $M$.

**Theorem**

Let $C$ be the integral weight part of parabolic category $\mathcal{O}$ for the complex general linear Lie superalgebra of block matrices with the $ij$-block of size $n_i \times n_j$ graded in parity $c_i + c_j \in \mathbb{Z}/2$, defined with respect to the standard parabolic of block upper triangular matrices. It admits a categorical action (just like the category $\mathcal{O}$ example given earlier) making it into a TPC of $M$.

The standard objects in $C$ are parabolic Verma modules, which identify with the monomial basis of $M$.
The existence of a Koszul grading implies further:

**Corollary**

The classes of the PIMs in $C$ correspond to Lusztig’s canonical basis for $M$.

This proves the super Kazhdan-Lusztig conjecture conjectured in 2003. A different proof was given already by Cheng, Lam and Wang.
Example: the oriented Brauer category

Assume $\mathfrak{g} = \mathfrak{sl}_{\infty}$ or $\hat{\mathfrak{sl}}_{p}$ according to $k$. Fix a parameter $\delta \in \mathbb{Z} \cdot 1_k$.

Let $\mathcal{OB}(\delta)$ be the free symmetric $k$-linear monoidal category generated by an object $\uparrow$ of dimension $\delta$ and its dual $\downarrow$.

Morphisms in $\mathcal{OB}(\delta)$ are $k$-linear combinations of diagrams. For example here is a morphism $\uparrow\uparrow\uparrow\downarrow \rightarrow \uparrow\uparrow\uparrow\downarrow\uparrow\downarrow$:

- Composition $\leftrightarrow$ vertical stacking
- Tensor product $\leftrightarrow$ horizontal stacking
- Bubbles $\leftrightarrow \delta$

The Karoubi envelope of $\mathcal{OB}(\delta)$ is Deligne’s category $\text{Rep}(GL_{\delta})$.

**Theorem (B.-Reynolds)**

The category of $k$-linear functors $\mathcal{OB}(\delta) \rightarrow \text{Vec}$ admits a categorical action making it a tensor product categorification of $V(-\omega_{-\delta}) \otimes V(\omega_0)$.

There are also cyclotomic oriented Brauer categories which categorify more general lowest weight tensored highest weight representations.

Webster has a different construction using cyclotomic quotients of $\mathcal{U}(\mathfrak{g})$. 