TPCs and Super Kazhdan-Lusztig conjecture
(tensor product categorification)
\[ \mathfrak{g}_0 \oplus \mathfrak{g}_1 \text{ 2/2-graded} \]
\[ \text{even} \quad \text{odd} \]
KL conjecture for Lie superalgebra

\[ \sigma_j = \sigma_j^{n \downarrow m} (C) \]

\[ \begin{array}{c|c}
\text{even} & \text{odd} \\
\hline
\text{odd} & \text{even} \\
\end{array} \]

+ supercommutator \[ [\cdot, \cdot] \]

Formulated in 2002 (B.)
(super duality)

Talk today about new proof by B.-Losev-Webster

× Application of uniqueness of TPCs (Losev's talk next!)
× Leads further to graded lift which is Koszul
× Parabolic analogs, non-standard Borels okay too
Super category \( \mathcal{O} \)

\[ U = U_\nu \oplus U_{\frac{m}{n}} \text{ natural representation} \]

Borel upper triangular matrices \( \begin{cases} (S_i, S_j) = 1 \\ (S_i, S_j) = -1 \end{cases} \) orthogonal

Cartan diagonal matrices

\( t^* \) has standard coordinates \( S_{1}, \ldots, S_{n}, S_{n+1}, \ldots, S_{n+m} \)

\( \Delta \) integral weights \( \mathbb{Z} S_{1} \oplus \cdots \oplus \mathbb{Z} S_{n+m} \)

Category \( \mathcal{O}_\mathbb{Z} \) left over \( \mathbb{T} \)

s.s. over \( t \), all weights integral

It's a highest weight category, weight poset \( (\Lambda, \leq) \)

Bruhat order

PIMs standard objects irreducibles

\( P(\lambda) \rightarrow \Delta(\lambda) \rightarrow L(\lambda), \lambda \in \Lambda \)

Verma module \( \mathcal{U}(\mathfrak{g}) \otimes \mathcal{C}_\lambda \)

\( K(\mathcal{O}_\Delta) \) complexified Groth group of \( \Delta \)-filtered modules

\( \mathbb{C} \)-vector space on basis \( \{ [\Delta(\lambda)] \} \)
Special projective functors

\[ O_2 \otimes F = \bigoplus_{i \in \mathbb{Z}} F_i = -\otimes U \]
\[ E = \bigoplus_{i \in \mathbb{Z}} E_i = -\otimes U^* \]
\( \{ \text{endofunctors} \} \)

\( x : F \to F \) action of \( \sum_{i \geq 1} (-1)^i e_{ij} \otimes e_{ji} \in \mathcal{O}_G \otimes \mathcal{O}_G \)

\( t : F^2 \to F^2 \) induced by tensor Casimir

\[ U \otimes U \to U \otimes U, u \otimes u \mapsto (-1)^{|l| |l|} u \otimes u \]

super braiding

These natural transformations induce an action of the degenerate affine Hecke algebra \( H_d \) on powers \( F^d \)

Then \( F_i = \text{gen. } i\text{-eigenspace of } x \)
\( E_i = \text{biadjoint} \)

\( \Rightarrow O_2 \) is an \( \mathfrak{sl}_\infty \)-categorification

(Chuang–Rouquier's axioms)

\[ \mathfrak{sl}_\infty / \mathfrak{sl}_2 \text{ Dynkin diagram } A_\infty \]

\[ \cdots \rightarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots \]
$\mathfrak{sl}_2$-Tensor product categorification

$\mathfrak{sl}_2$ Chevalley generators $e_i, f_i, h_i$ (i.e $\mathbb{Z}$)

$V_2$ natural module on basis $\{v_i \mid i \in \mathbb{Z}\}$

$W_2$ dual module on basis $\{w_i \mid i \in \mathbb{Z}\}$

**Theorem** $O_2$ is an $\mathfrak{sl}_2$-TPC of $V_2^\otimes n \otimes W_2^\otimes m$

(Losev-Webster's axioms)

**Meaning:**

- Its $h/w$ category acts on set $\Lambda \cong \mathbb{Z}^{n+m}$
  
  $\lambda \mapsto (\lambda_1, \ldots, \lambda_{n+m})$

  $\lambda_i = (\lambda + s, s_i)$

- Its $\mathfrak{sl}_2$-categorification

- The functors $E_i, F_i$ preserve $O_2^\Lambda$

- $K(O_2^\Lambda) \cong V_2^\otimes n \otimes W_2^\otimes m$

  $[\Delta(\lambda)] \mapsto v_\lambda = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n} \otimes w_{\lambda_{n+1}} \otimes \cdots \otimes w_{\lambda_{n+m}}$

- $\mathfrak{gl} : \mathfrak{sl}_2$

- **Theorem** (SKL) $[P(\lambda)] \.mapsto b_\lambda$ Lustig's canonical basis
Lusztig's canonical basis

Quantized enveloping algebra $U_q^{\text{sl}_2} \big/ \mathcal{O}(q)$

$\Delta(\hat{f}_i) = 1 \otimes \hat{f}_i + \hat{f}_i \otimes k_i$  $\hat{e}_i, \hat{f}_i, \hat{k}_i$ deformed generators

Theorem (essentially Lusztig) $V_{\hat{\lambda}} \otimes W_{\hat{\mu}}$ admits a unique bar involution $\Psi$ s.t.

- $\Psi$ commutes with $\hat{e}_i$ and $\hat{f}_i$, antilinear $q \rightarrow q^{-1}$
- $\Psi(\hat{V}_\lambda) = \hat{V}_\lambda + (2 \mathbb{Z}[q, q^{-1}] \text{-comb of } \hat{V}_\mu, \mu > \lambda)$

Then $b_\lambda$ is $b_\lambda$ specialized at $q=1$ where

- $\Psi(b_\lambda) = b_\lambda$
- $b_\lambda = \hat{V}_\lambda + (q \mathbb{Z}[q] \text{-comb of } \hat{V}_\mu, \mu > \lambda)$

SKL says: some probable KL polynomial $\in \mathbb{N}[q]$

If $b_\mu = \sum d_{\lambda \mu}(q) \hat{V}_\lambda$

Then $[\Delta(\lambda) : L(\mu)] = \sum d_{\lambda \mu}(1)$

composition multiplicity

of Verma module
New proof

Take $I \subset \mathbb{Z}$ a finite interval

$$\mathfrak{sl}_I \hookrightarrow \mathfrak{sl}_2$$

$$V_I \hookrightarrow V_2$$

$$W_I \hookrightarrow W_2$$

Show $O_2$ has subquotient $O_I$ which is an $\mathfrak{sl}_I$-TPC of $V_I \otimes W_I$.

Then appeal to Losev-Webster's uniqueness theorem

Get $O_I \cong$ known classical category $O$ (parabolic)

$\Rightarrow$

Beilinson-Ginzburg-Soergel showed this category is Koszul

$\Rightarrow$ SKL follows from classical KL + LW uniqueness
Graded lift and Koszulity.

With substantially more work we have shown the BAS graduijs on each $O_1$ lift to $O_2$.

**Theorem** $O_2$ has a graded lift $\hat{O}_2$ so that

$$K(\hat{O}_2) \cong V_2^\otimes n \otimes \omega_2^\otimes m.$$

Graded TPC

Unique

Koszul

$O_2 \cong \text{mod}-A$

$$A = \bigoplus_{x \in \Lambda} \text{Hom} (P(x), P(y))$$

$\hat{O}_2 := \text{grmod}-A$

Admits unique Koszul grading

A here is locally finite algebra — blocks in $O_2$ are infinite.
Double centralizer property.

\[ C = \bigoplus_{\lambda \in \Lambda} \text{Hom}_\mathcal{O}(P(\lambda), P(\lambda)) \]

To get graded let sufficient to show
\[ C \text{ admits grading as do all } WP(\lambda) \]

\[ 0 \rightarrow \text{mod-} C \]

\[ M \rightarrow \bigoplus \text{Hom}(P(\lambda), M) \]

Quotient functor

is fully faithful on projectives

\[ A = \bigoplus_{\lambda \in \Lambda} \text{Hom}_\mathcal{O}(WP(\lambda), WP(\lambda)) \]

We cannot describe \( C \) explicitly (analog of covariant algebra in Soergel's work).

Instead we proceed indirectly, replacing \( \text{mod-} C \) with another abelian category of stable modules over a tower of cyclotomic QHA's.

END