2007 Boot Camp, Problem Set 1

One of the primary goals of boot camp will be to familiarize you with some techniques for solving the sorts of problems that will occur in your 500/600-level classes. These same sorts of problems will occur on your qualifying exams, so it's good to get comfortable with them early, and they really are helpful for learning to do research in the end.

There are four essential exercise types that occur in mathematics courses: define/state, true/false, compute, and prove: Define/state problems require some level of memorization when one is new to a subject, but rapidly become trivial as one's understanding of why a particular definition is made or theorem is true increases. Our purpose with these problems will be two-fold: to give you an idea which common definitions and theorems you need to review and to try to make the reason some of these definitions and theorems are the way they are clearer.

True/false problems tend to be the most difficult of the problem types when one is unfamiliar with a subject as they test the sharpness of one's knowledge – a slightly weakened condition makes something that seemed true become false, or something that seems too good to be true if one isn't thinking cleverly turns out to be so. Our purpose with these problems will be to get you into the habit of writing down examples for theorems you run across and counterexamples for those same theorems when conditions are relaxed. If I had one habit that I could have instilled in myself before starting classes, this would be it. To answer true/false problems: if true, write down an example and a sketch of the proof if you can; if false, write down two non-trivial counterexamples. (Here, non-trivial means "not the first thing that comes to mind.")

Computations ground us back to reality. It's going to be very tempting to forget that all of the theorems we spend so much time proving exist so we can actually do computations. Taking a moment to apply the theorems gives an opportunity to tie all of our knowledge back together and see it as a whole.

And lastly, proofs are fairly self-explanatory, as we all hope to be doing them for a living some day.

This first problem set is going to focus on Linear Algebra, though will contain a few basic problems of the type you might see at the beginning of 500-level courses, just to spice things up. Linear Algebra is generally assumed to be background knowledge by 500/600-level instructors at a level most undergraduates don't necessarily see or recall. Take some time to work through these problems on your own and we will proceed depending on the average level of progress and number of questions when we get together for the first time. Much of this will be used in the other problem sets and we will see quite a bit of other linear algebra along the way. If you have questions, let me know!

1. Linear Algebra

Here, unless otherwise noted, F is a field, V and W are F vector spaces.

(1) Define...
  (a) a field.
  (b) an F vector space.
  (c) a linear map f between vector spaces.
  (d) a basis for an F vector space.
  (e) similar matrices
  (f) characteristic and minimal polynomials of a linear map
  (g) eigenvalues and eigenvectors of linear maps
  (h) diagonalizable linear maps
  (i) Jordan Normal Form of a matrix
(2) Give as many examples of (types of) fields and vector spaces over those fields as you can.
(3) Let V be a finite dimensional vector space. Explain what it means for a matrix A to represent a linear map $f : V \to V$. Prove that matrices A and B represent f if and only if A and B are similar.
(4) Using the last problem, explain what is meant by a property of a linear map being *basis invariant*. The basis invariant properties of a vector space/general mathematical object tend to be those that matter.
(5) Let V be the set of all real polynomials in t with degree less than n (written by abuse of ring notation as $V = \mathbb{R}[t]/(t^n)$)
  (a) Show that V an $\mathbb{R}$ vector space.

  - (b) Differentiation is a linear map $\frac{\mathrm{d}}{\mathrm{d}t} : V \to V$. Exhibit two different matrices that represent $\frac{d}{dt}$.
  - (c) Integration is not a linear map $\int \mathrm{d}t : V \to V$. Why not? How can this problem be rectified? This being done, can you represent this map using a matrix?
- (6) Let $F = \mathbb{R}$. Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & -8 \\ -\frac{1}{3} & 2 & \frac{8}{3} \\ 0 & 0 & -1 \end{pmatrix}$$

  - (a) Compute the determinant of A.
  - (b) Calculate the characteristic polynomial $p(t)$ and the minimal polynomial $m(t)$ for A.
  - (c) Find the eigenvalues and eigenvectors of A.
  - (d) Is A diagonalizable? If so, find a basis in which it is diagonal.
  - (e) Find the Jordan Normal Form of A. (If A is diagonalizable, you're already done!)
- (7) Do the last problem with $F = \mathbb{Z}/11\mathbb{Z}$ and $F = \mathbb{Z}/2\mathbb{Z}$.
- (8) If V is a vector space, we define the *dual* of V

$$V^* = \{f : V \to F | f \text{ linear}\}$$

  - (a) Show that V^* is an F vector space
  - (b) If V is finite dimensional with basis $\{v_1, v_2, \ldots, v_n\}$, define $f_i \in V^*$ by $f_i(v_j) = \delta_i j$, where $\delta_i j$ is the Kronecker delta function. Show that $\{f_1, f_2, \ldots, f_n\}$ forms a basis for V^* called the *dual basis*. Conclude that $\dim V = \dim V^*$.
  - (c) Show that $\mathbb{R}[t]$, the set of all polynomials with real coefficients, is an $\mathbb{R}$ vector space with basis $\{t^i | i \in \mathbb{N}\}$. In particular, this is not a finite dimensional vector space.
  - (d) Show that the "dual basis" to the vector space basis for $\mathbb{R}[t]$ given above does not span $\mathbb{R}[t]^*$. In general, finite dimensionality is required to make the dual basis work.
- (9) True or False:
  - (a) Every linear operator over a real vector space is diagonalizable.
  - (b) Every invertible linear operator over a real vector space is diagonalizable.
  - (c) Every invertible linear operator over a complex vector space is diagonalizable.
  - (d) Each of the above, with "is diagonalizable" replaced by "can be put in Jordan Normal Form".
  - (e) There is precisely one vector space up to isomorphism of each finite dimension over any field F.
- (10) Let B be a 4×4 matrix over $\mathbb{C}$ with characteristic polynomial $(x^2 + 1)(x - 2)^3$. Find all possible Jordan normal forms of B. What happens if we consider B as a matrix over $\mathbb{R}$?

2. 500 ANALYSIS

- (1) Prove that sequences in $\mathbb{R}$ are convergent if and only if they are Cauchy.
- (2) Let f and g be continuous real valued functions on $\mathbb{R}$, $x_0 \in \mathbb{R}$ a point such that $f(x_0) > g(x_0)$. Prove that there exists an interval (a, b) such that $x_0 \in (a, b)$ and $f(x) > g(x)$ for all $x \in (a, b)$.
- (3) Let (s_n) and (t_n) be sequences of real numbers.
  - (a) State the definition of $\limsup s_n$ and $\liminf s_n$.
  - (b) Prove that $\limsup s_n \geq \liminf s_n$.
  - (c) What can you say about (s_n) if $\limsup s_n = \liminf s_n$? Prove your assertion.
  - (d) Prove that $\liminf(s_n + t_n) \geq \liminf s_n + \liminf t_n$.

3. 500 ALGEBRA

- (1) Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.
- (2) True or False: There is a group G which is a union of two proper subgroups.
- (3) True or False: There is a group G which is a union of three proper subgroups.
- (4) Prove that a group which has only a finite number of subgroups is finite.
- (5) Let H and K be subgroups of a finite group. Prove that $|HK| = \frac{|H||K|}{|H \cap K|}$.

4. 500 TOPOLOGY

All of these fit just as well in the 500 Analysis section. Here, assume we're talking about a metric space X.

1. Prove that the complement of an open set is closed.
2. Prove that the complement of a closed set is open.
3. Prove that a compact subset of a metric space is closed.
4. Prove that closed subsets of compact sets are compact.
5. Prove that closed, bounded subsets of $\mathbb{R}$ are compact.