Advances in Number Theory and Random Matrix Theory

Random Matrix Theory and Heights of Polynomials

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PIMS, SFU, UBC
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Ginibre’s Ensembles

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- Labeled GinOE, GinUE and GinSE respectively.

Ginibre’s goal was to compute the joint eigenvalue probability density functions (JPDF) and the correlation functions for these ensembles.
GinOE

GinOE is defined to be $\mathbb{R}^{N \times N}$ together with the probability measure $\nu$ given by

$$
\nu(S) := \frac{1}{\nu(\mathbb{R}^{N \times N})} \int_S \exp \left\{ -\frac{1}{2} \text{Tr}(X^TX) \right\} \, d\lambda(X),
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where $\lambda$ is Lebesgue measure on $\mathbb{R}^{N \times N}$.
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GinOE is the most complicated of Ginibre’s ensembles since $\mathbb{R}$ is not algebraically closed.

The eigenvalues of $X \in \mathbb{R}^{N \times N}$ are partitioned into real and complex conjugate pairs.
The space of eigenvalues of $\mathbb{R}^{N \times N}$ can be written as the disjoint union

$$\bigcup_{(L,M)} \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M,$$

where the union is over all pairs $(L, M)$ such that $L + 2M = N$. 
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In order to compute the JPDF we need to find the partial JPDFs, $P_{L,M} : \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M \rightarrow \mathbb{R}$ for each pair $(L, M)$. 
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$$
P_{N,0}(\alpha) = C_N^{-1} \frac{1}{N!} \left\{ \prod_{\ell=1}^{N} e^{-\alpha_\ell^2 / 2} \right\} \left\{ \prod_{j<k} |\alpha_k - \alpha_j| \right\}.
$$
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$$
The Partial JPDFs

Given $\gamma \in \mathbb{C}^N$ let $V_{\gamma}$ be the $N \times N$ Vandermonde determinant in the coordinates of $\gamma$. 
The Partial JPDFs

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$$V\gamma := \begin{bmatrix} 1 & 1 & 1 \\ \gamma_1 & \gamma_2 & \gamma_N \\ \gamma_1^2 & \gamma_2^2 & \gamma_N^2 \\ \vdots & \vdots & \vdots \\ \gamma_1^{N-1} & \gamma_2^{N-1} & \gamma_N^{N-1} \end{bmatrix}$$
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\end{bmatrix}$$

Given $\alpha \in \mathbb{C}^L$ and $\beta \in \mathbb{C}^M$ we define

$$\Delta(\alpha, \beta) := \det V\gamma$$

where

$$\gamma := (\alpha_1, \ldots, \alpha_L, \overline{\beta_1}, \beta_1, \ldots, \overline{\beta_M}, \beta_M).$$
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**Conjecture 1 (S-2005).** There exist functions $w_1 : \mathbb{R} \to \mathbb{R}$ and $w_2 : \mathbb{C} \to \mathbb{R}$ such that

$$P_{L,M}(\alpha, \beta) \propto \frac{|\Delta(\alpha, \beta)|}{L!M!}$$

$$\times \left\{ \prod_{\ell=1}^L w_1(\alpha_\ell) \right\} \left\{ \prod_{m=1}^M w_2(\beta_m) \right\},$$
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**Theorem 1 (Lehmann & Sommers 1991, Edelman 1997).** There exist functions $w_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $w_2 : \mathbb{C} \rightarrow \mathbb{R}$ such that

$$P_{L,M}(\alpha, \beta) = C_N^{-1} \frac{|\Delta(\alpha, \beta)|}{L!M!} \times \left\{ \prod_{\ell=1}^{L} w_1(\alpha_\ell) \right\} \left\{ \prod_{m=1}^{M} w_2(\beta_m) \right\},$$

where $C_N = 2^{N(N+1)/4} \prod_{n=1}^{N} \Gamma(n/2)$. 

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The Partial JPDFs

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$$P_{L,M}(\alpha, \beta) = \mathcal{C}_N^{-1} \frac{|\Delta(\alpha, \beta)|}{L!M!} \times \left\{ \prod_{\ell=1}^L e^{-\alpha_{\ell}^2/2} \right\} \left\{ \prod_{m=1}^M \text{erfc} \left( \sqrt{2} |\text{Im}\beta_m| \right) e^{-(\beta_m^2 + \overline{\beta}_m^2)/2} \right\},$$

where $\mathcal{C}_N = 2^{N(N+1)/4} \prod_{n=1}^N \Gamma(n/2)$. 
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We may simplify this by setting $w : \mathbb{C} \rightarrow \mathbb{R}$, where

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Since \( \text{erfc}(0) = 1 \),

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The two main difficulties in working with GinOE are now apparent:

1. The decomposition of the space of eigenvalues.
2. GinOE is a $\beta = 1$ ensemble.
A function $\Psi : \mathbb{R}^{N \times N} \rightarrow \mathbb{R}$ will be called a multiplicative class function if

- $\Psi(A X A^{-1}) = \Psi(X)$ for all invertible $A \in \mathbb{R}^{N \times N}$. 
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- $\Psi(AXA^{-1}) = \Psi(X)$ for all invertible $A \in \mathbb{R}^{N \times N}$.
- There exists a function $\psi : \mathbb{C} \to \mathbb{R}$ such that if $D$ is a diagonal matrix with entries $\gamma_1, \gamma_2, \ldots, \gamma_N$ then

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Here we will provide method for determining the ensemble averages of multiplicative class functions.
Multiplicative Class Functions

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$$\langle \Psi \rangle := \frac{1}{(\sqrt{2\pi})^{N^2}} \int_{\mathbb{R}^{N \times N}} \Psi(X) \exp \left\{ -\frac{1}{2} \text{Tr}(X^TX) \right\} d\lambda(X).$$
Averages over GinOE

\( \lambda \)-almost every matrix is diagonalizable, and hence \( \langle \Psi \rangle \) is uniquely determined by \( \psi \).
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\( \lambda \)-almost every matrix is diagonalizable, and hence \( \langle \Psi \rangle \) is uniquely determined by \( \psi \). Thus, from the partial JPDFs,

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\langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \right\} \\
\quad \times |\Delta(\alpha, \beta)| \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta),
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- $\varphi(\gamma) = w(\gamma) \psi(\gamma)$,
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- $\lambda_{2M}$ is Lebesgue measure on $\mathbb{C}^M$. 

PIMS, SFU, UBC
Skew-symmetric inner products

We introduce two skew-symmetric bilinear forms associated to $\Psi$:

$$\langle P, Q \rangle_\mathbb{R} := \int_{\mathbb{R}^2} \varphi(\alpha_1)\varphi(\alpha_2) \, P(\alpha_1)Q(\alpha_2) \, \text{sgn}(\alpha_2 - \alpha_1) \, d\alpha_1 \, d\alpha_2,$$
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and,

$$\langle P, Q \rangle_{\mathbb{C}} := -2i \int_{\mathbb{C}} \varphi(\beta) \varphi(\overline{\beta}) P(\overline{\beta}) Q(\beta) \, \text{sgn}(\text{Im}(\beta)) \, d\lambda_2(\beta).$$
We introduce two skew-symmetric bilinear forms associated to \( \Psi \):

\[
\langle P, Q \rangle_R := \int_{\mathbb{R}^2} \varphi(\alpha_1)\varphi(\alpha_2) \ P(\alpha_1)Q(\alpha_2) \ sgn(\alpha_2 - \alpha_1) \ d\alpha_1 \ d\alpha_2,
\]

and,

\[
\langle P, Q \rangle_C := -2i \int_{\mathbb{C}} \varphi(\beta)\varphi(\overline{\beta}) \ P(\overline{\beta})Q(\beta) \ sgn(\text{Im}(\beta)) \ d\lambda_2(\beta).
\]

By construction,

\[
\langle Q, P \rangle_R = -\langle P, Q \rangle_R \quad \text{and} \quad \langle Q, P \rangle_C = -\langle P, Q \rangle_C.
\]
Theorem 2 (S-2006). Let $N = 2J$, and let

$$\mathbf{P} = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\}$$

be a set of monic polynomials with $\deg P_n = n - 1$. Then,

$$\langle \Psi \rangle = C_N^{-1} \text{Pf } U_\mathbf{P},$$

the Pfaffian of $U_\mathbf{P}$, where $U_\mathbf{P}$ is the $N \times N$ matrix where,

$$U_\mathbf{P}[j, k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}.$$
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$$U_P[j, k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}.$$ 

When $N$ is odd the matrix $U_P$ must be modified.
Corollary 2. Let $\langle P, Q \rangle = \langle P, Q \rangle_R + \langle P, Q \rangle_C$, and let $Q = \{Q_1, Q_2, \ldots, Q_N\}$ be a set of monic polynomials specified by

$$\langle Q_{2k-1}, Q_{2j} \rangle = -\langle Q_{2j}, Q_{2k-1} \rangle = \delta_{kj} M_j$$

and

$$\langle Q_{2j}, Q_{2k} \rangle = \langle Q_{2j-1}, Q_{2k-1} \rangle = 0,$$

Then,

$$\langle \Psi \rangle = c_N^{-1} \prod_{j=1}^{J} M_j.$$
Heights of Polynomials

Let \( f(x) \in \mathbb{C}[x] \) be given by

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\mu(f) = |a_N| \prod_{n=1}^{N} \max\{1, |\gamma_n|\}
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By Jensen’s formula we also have

\[
\mu(f) = \exp \left\{ \int_0^1 \log |f(e^{2\pi i \theta})| \, d\theta \right\}.
\]
Mahler Measure

Mahler measure can be interpreted as:
• The geometric mean of $f$ on the unit circle.
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Mahler measure is:

• continuous on finite dimensional subspaces of $\mathbb{C}[x]$
• multiplicative: $\mu(fg) = \mu(f)\mu(g)$,
• absolutely homogeneous: $\mu(kf) = |k|\mu(f)$,
• positive definite: $\mu(f) = 0$ iff $f = 0$. 
Lehmer’s Problem

Number Theorists are interested in the range of values of Mahler measure on $\mathbb{Z}[x]$. 
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The following problem arises immediately. If \( \epsilon \) is a positive quantity to find a polynomial of the form

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f(x) = x^r + a_1 x^{r-1} + \ldots + a_r \]

where the \( a \)'s are integers, such that the absolute value of those roots of \( f \) which lie outside the unit circle lies between 1 and \( 1 + \epsilon \). (D.H. Lehmer 1933)
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In the same paper D.H. Lehmer states,

We have not made an examination of all 10th degree symmetric polynomials, but a rather intensive search has failed to reveal a better polynomial than

$$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1, \quad \mu = 1.176 \ldots$$
Smyth’s Theorem

A polynomial $f$ is said to be *reciprocal* if

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A polynomial \( f \) is said to be \textit{reciprocal} if

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f(x) = x^{\deg f} f\left(\frac{1}{x}\right).
\]

- The coefficients of a reciprocal polynomial are palindromic.
- If \( f(\alpha) = 0 \) then \( f(\alpha^{-1}) = 0 \).
Smyth’s Theorem

A polynomial $f$ is said to be reciprocal if

$$f(x) = x^\deg f f\left(\frac{1}{x}\right).$$

- The coefficients of a reciprocal polynomial are palindromic.
- If $f(\alpha) = 0$ then $f(\alpha^{-1}) = 0$.

**Theorem 3 (Smyth 1971).** If $g$ is an irreducible non-reciprocal polynomial in $\mathbb{Z}[x]$, and $g(x) \nmid x(x - 1)$, then

$$\mu(g) \geq \mu(x^3 - x - 1) = 1.324 \ldots$$
Let $J$ be the integer part of $\frac{(N - 1)}{2}$.

**Theorem 4 (Chern & Vaaler 2001).** As $T \to \infty$,

$$\# \{ f \in \mathbb{Z}[x] : \deg f = N, \mu(f) \leq T \} = \left\{ \frac{2^{N-J+1}(N+1)^J}{J!} \prod_{j=1}^{J} \left( \frac{2j}{2j+1} \right)^{N+1-2j} \right\} T^{N+1} + O(T^N).$$
The Range of Mahler Measure

Let $J$ be the integer part of $(N - 1)/2$.

**Theorem 5 (Chern & Vaaler 2001).** As $T \to \infty$,

$$
    \#\{ f \in \mathbb{Z}[x] : \deg f = N, \mu(f) \leq T \} = 
    \left\{ \frac{2^{N-J+1}(N + 1)^J}{J!} \prod_{j=1}^{J} \left( \frac{2j}{2j + 1} \right)^{N+1-2j} \right\} T^{N+1} + O(T^N).
$$

**Theorem 5 (S- 2005).** As $T \to \infty$,

$$
    \#\{ f \in \mathbb{Z}[x] : \deg f = N, f \text{ reciprocal}, \mu(f) \leq T \} = 
    \left\{ \frac{2^{2J+3}(N + 1)^N}{N!} \prod_{n=1}^{N} \left( \frac{2n}{2n + 1} \right)^{N+1-n} \right\} T^{J+1} + O(T^J).
$$
The Connection with GinOE

Recall our main result,

\[ \langle \Psi \rangle = C_N^{-1} \text{Pf} \, U_P, \]

where \( U_P \) is the \( N \times N \) matrix,

\[ U_P[j,k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}. \]
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\[ U_P[j,k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}. \]

The inspiration for this result came from the proof that as \( T \to \infty \),

\[ \# \{ f \in \mathbb{Z}[x] : \deg f = N, f \text{ reciprocal}, \mu(f) \leq T \} = \]

\[ \left\{ \frac{2^{2J+3}(N + 1)^N}{N!} \prod_{n=1}^{N} \left( \frac{2n}{2n + 1} \right)^{N+1-n} \right\} T^{J+1} + O(T^{J}). \]
A function $\Phi : \mathbb{C}[x] \to [0, \infty)$ is called a multiplicative distance function if:

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There exists a *root function* $\phi : \mathbb{C} \rightarrow (0, \infty)$ such that

$$
\Phi : a_N \prod_{n=1}^{N} (x - \gamma_n) \leftrightarrow |a_N| \prod_{n=1}^{N} \phi(\gamma_n).
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Multiplicative Distance Functions

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$$\Phi : a_N \prod_{n=1}^{N} (x - \gamma_n) \mapsto |a_N| \prod_{n=1}^{N} \phi(\gamma_n).$$

Moreover, as $|\gamma| \to \infty$, $\phi(\gamma) \sim |\gamma|$. 
Multiplicative Distance Functions

- Multiplicative distance functions restricted to finite dimensional subspaces of $\mathbb{C}[x]$ are distance functions in the sense of the geometry of numbers.
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• As such, they satisfy all the axioms of vector norms except the triangle inequality.
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- Multiplicative distance functions restricted to finite dimensional subspaces of $\mathbb{C}[x]$ are distance functions in the sense of the geometry of numbers.
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- The unit balls are compact but not convex — they are *star bodies*.
- If $\phi(\gamma) = \max\{1, |\gamma|\}$ then $\Phi$ is Mahler measure.
- The algebra of reciprocal polynomials is isomorphic to the algebra $\mathbb{C}[x + 1/x]$, if

$$
\phi(\gamma) = \mu(x + 1/x - \gamma) = \max \left\{ \left| \frac{\gamma \pm \sqrt{\gamma^2 - 4}}{2} \right| \right\},
$$

the $\Phi$ is the *reciprocal Mahler measure*. 

PIMS, SFU, UBC
Star Bodies

Given \( a \in \mathbb{C}^{N+1} \), let

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a(x) := \sum_{n=1}^{N+1} a_n x^{N+1-n}.
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We then define

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By the absolute homogeneity of $\Phi$,

$$\{a \in \mathbb{R}^{N+1} : \Phi(a) \leq T\} = T U_N.$$
Asymptotic Estimates

When $T$ is large, the volume of $T \cup_N$ is a good estimate for the number of lattice points that it contains.
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When $T$ is large, the volume of $TU_N$ is a good estimate for the number of lattice points that it contains.

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$$\#\{f \in \mathbb{Z}[x]: \deg f = N, \mu(f) \leq T\} = \left\{ \frac{2^{N-J+1}(N+1)^J}{J!} \prod_{j=1}^{J} \left( \frac{2j}{2j+1} \right)^{N+1-2j} \right\} T^{N+1} + O(T^N).$$
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Recall, when $J = \lfloor (N - 1)/2 \rfloor$,

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$\text{vol}(U_N(\mu))$
We define the monic restriction of $\Phi$ to be the function
$\tilde{\Phi} : \mathbb{C}^N \rightarrow (0, \infty)$ given by

$$\tilde{\Phi}(b) = \Phi \left( x^N + \sum_{n=1}^{N} b_n x^{N-n} \right).$$
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$$

The Mellin transform of $f_N(\xi)$ is then

$$
\hat{f}_N(s) = \int_0^\infty \xi^{-s} f_N(\xi) \frac{d\xi}{\xi}.
$$
The Volume of $\mathcal{U}_N$

Theorem 6.

$$\text{vol}(\mathcal{U}_N) = 2\widehat{f}_N(N + 1).$$
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$$\text{vol}(\mathcal{U}_N) = \int_{-\infty}^{\infty} \lambda_N \{ \mathbf{b} \in \mathbb{R}^N : \Phi(y, \mathbf{b}) \leq 1 \} \, dy$$
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change of variables:

$$y = \frac{1}{\xi} \quad dy = -\frac{d\xi}{\xi^2}.$$
The Volume of $\mathcal{U}_N$

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\text{vol}(\mathcal{U}_N) = 2 \int_0^\infty \xi^{-N-1} \lambda_N \left\{ c \in \mathbb{R}^N : \tilde{\Phi}(c) \leq \xi \right\} \frac{d\xi}{\xi}
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PIMS, SFU, UBC
The Volume of $\mathcal{U}_N$

Theorem 6.

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Moment Functions

\[ \tilde{f}_N(s) = \int_0^\infty \xi^{-s} f_N(\xi) \frac{d\xi}{\xi}. \]
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$\hat{f}_N(s) = \int_0^\infty \xi^{-s-1} f_N(\xi) \, d\xi.$

integration by parts:

$u(\xi) = f_N(\xi)$

$dv = \xi^{-s-1} \, d\xi$
\[ \hat{f}_N(s) = \int_0^\infty \xi^{-s-1} f_N(\xi) \, d\xi. \]

integration by parts: (Lebesgue-Stieltjes)

\[
\begin{align*}
  u(\xi) &= f_N(\xi) \\
v &= \frac{1}{s} \xi^{-s} \\
du &= df_N(\xi) \\
dv &= \xi^{-s-1} \, d\xi
\end{align*}
\]
Moment Functions

\[
\hat{f}_N(s) = \frac{-f_N(\xi)\xi^{-s}}{s} \bigg|_0^\infty + \frac{1}{s} \int_0^\infty \xi^{-s} df_N(\xi)
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Moment Functions

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Recall that \( df_N = d\nu \) where \( \nu \) is the Borel measure on \((0, \infty)\) for which

\[ \nu(a, b] = f_N(b) - f_N(a) \quad 0 < a < b < \infty. \]
Moment Functions

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Recall that \( df_N = d\nu \) where \( \nu \) is the Borel measure on \((0, \infty)\) for which

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Moment Functions

\[ \widetilde{f}_N(s) = \frac{1}{s} \int_0^\infty \xi^{-s} df_N(\xi) \]

Recall that \( df_N = d\nu \) where \( \nu \) is the Borel measure on \((0, \infty)\) for which

\[ \nu(a, b] = \lambda_N \left\{ a \in \mathbb{R}^N : a < \tilde{\Phi}(a) \leq b \right\} \quad 0 < a < b < \infty. \]

Thus we may replace the integral over \((0, \infty)\) with an integral over \(\mathbb{R}^N\).
Moment Functions

\[ \hat{f}_N(s) = \frac{1}{s} \int_{\mathbb{R}^N} \tilde{\Phi}(b)^{-s} \, d\lambda_N(b) \]
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We define the degree \( N \) real moment function of \( \Phi \) to be the function of the complex variable \( s \) given by

\[ F_N(s) = \int_{\mathbb{R}^N} \tilde{\Phi}(b)^{-s} d\lambda_N(b). \]
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\( F_N(s) \) converges to an analytic function of \( s \) in the half plane \( \Re(s) > N \).
Examples of Moment Functions

As $T \to \infty$,

$$\# \{ f \in \mathbb{Z}[x] : \deg f = N, \Phi(f) \leq T \} = \operatorname{vol}(\mathcal{U}_N)T^{N+1} + O(T^N).$$
Examples of Moment Functions

As $T \to \infty,$

$$\#\{f \in \mathbb{Z}[x] : \deg f = N, \Phi(f) \leq T\} = \frac{2F_N(N+1)}{N+1}T^{N+1} + O(T^N).$$
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Theorem 8 (Chern & Vaaler 2001). Let $J = \lfloor (N - 1)/2 \rfloor$, then

$$F_N(\mu; s) = 2^N \left\{ \prod_{k=1}^{J} \left( \frac{2k}{2k+1} \right)^{N-2k} \right\} \left\{ \prod_{j=0}^{J} \frac{s}{s - (N - 2j)} \right\}.$$
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Theorem 8 (S- 2005). Let $\rho$ be the reciprocal Mahler measure. Then,

$F_N(\rho; s) = \frac{2^N}{N!} \left\{ \prod_{n=1}^{N} \left( \frac{2n}{2n - 1} \right)^{N+1-n} \right\} \left\{ \prod_{j=0}^{J} \frac{s^2}{s^2 - (N - 2j)^2} \right\}$.
A Change of Variables

Next we will use the multiplicativity of $\Phi$ to rewrite $F_N(s)$ in terms of the roots of polynomials.
A Change of Variables

The space of roots of degree $N$ polynomials in $\mathbb{R}[x]$ can be written as the disjoint union

$$\bigcup_{(L,M)} \mathbb{R}^L \times (\mathbb{C} \setminus \mathbb{R})^M,$$

where the union is over all pairs $(L, M)$ such that $L + 2M = N$. 
A Change of Variables

Given \((L, M)\), define

\[ E_{L,M} : \mathbb{R}^L \times \mathbb{C}^M \rightarrow \mathbb{R}^N \]

by \( E_{L,M}(\alpha, \beta) := b \) where

\[ x^N + \sum_{n=1}^{N} b_n x^{N-n} = \prod_{\ell=1}^{L} (x - \alpha_\ell) \prod_{m=1}^{M} (x - \beta_m)(x - \overline{\beta_m}). \]
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\]

The degree of the map \(E_{L,M}\) is \(2^M M! L!\), and the images of the various \(E_{L,M}\) are disjoint (except for a set of \(\lambda_N\)-measure 0).
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by \( E_{L,M}(\alpha, \beta) := b \) where

\[
x^N + \sum_{n=1}^{N} b_n x^{N-n} = \prod_{\ell=1}^{L} (x - \alpha_\ell) \prod_{m=1}^{M} (x - \beta_m)(x - \overline{\beta_m}).
\]

The degree of the map \( E_{L,M} \) is \( 2^M M! L! \), and the images of the various \( E_{L,M} \) are disjoint (except for a set of \( \lambda_N \)-measure 0).

Moreover,

\[
\tilde{\Phi}(E_{L,M}(\alpha, \beta)) = \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta_m}).
\]
A Change of Variables

\[ F_N(s) = \int_{\mathbb{R}^N} \tilde{\Phi}(b)^{-s} d\lambda_N(b) \]
A Change of Variables

\[ F_N(s) = \sum_{(L,M)} \frac{1}{2^M M! L!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \tilde{\Phi}(E_{L,M}(\alpha, \beta))^{-s} \times \text{Jac } E_{L,M}(\alpha, \beta) \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta). \]
A Change of Variables

\[ F_N(s) = \sum_{(L,M)} \frac{1}{2^M M! L!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \Phi(E_{L,M}(\alpha, \beta))^{-s} \times \text{Jac } E_{L,M}(\alpha, \beta) \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta). \]
A Change of Variables

\[ F_N(s) = \sum_{(L,M)} \frac{1}{2^M M! L!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m) \phi(\overline{\beta_m}) \right\}^{-s} \times \text{Jac} E_{L,M}(\alpha, \beta) \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta). \]
A Change of Variables

\[ F_N(s) = \sum_{(L,M)} \frac{1}{2^M M! L!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m) \phi(\overline{\beta_m}) \right\}^{-s} \times \text{Jac } E_{L,M}(\alpha, \beta) \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta). \]

Lemma 8.

\[ \text{Jac } E_{L,M}(\alpha, \beta) = 2^M |\Delta(\alpha, \beta)|. \]
A Change of Variables

\[ F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m) \phi(\overline{\beta_m}) \right\}^{-s} \]

\[ \times |\Delta(\alpha, \beta)| d\lambda_L(\alpha) d\lambda_{2M}(\beta). \]
The Connection

\[ F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^L \phi(\alpha_\ell) \prod_{m=1}^M \phi(\beta_m)\phi(\overline{\beta}_m) \right\}^{-s} \times |\Delta(\alpha, \beta)| d\lambda_L(\alpha) d\lambda_{2M}(\beta). \]
The Connection

\[ F_N(s) = \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \phi(\alpha_\ell) \prod_{m=1}^{M} \phi(\beta_m)\phi(\overline{\beta}_m) \right\}^{-s} \]
\[ \times |\Delta(\alpha, \beta)| \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta). \]

And recall that

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta}_m) \right\} \]
\[ \times |\Delta(\alpha, \beta)| \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta), \]

where \( \varphi(\gamma) = \psi(\gamma) e^{-\gamma^2/2} \left\{ \text{erfc}(\sqrt{2}|\text{Im}(\gamma)|) \right\}^{1/2}. \)
The Idea of the Proof

We wish to prove

\[ \langle \Psi \rangle = C_N^{-1} \text{Pf } U_P, \]

where

\[ U_P[j, k] := \langle j, k \rangle_R + \langle j, k \rangle_C. \]
The Idea of the Proof

We wish to prove

$$\langle \Psi \rangle = C_N^{-1} \text{Pf } U_{P},$$

where

$$U_{P}[j, k] := \langle P_j, P_k \rangle_{\mathbb{R}} + \langle P_j, P_k \rangle_{\mathbb{C}}.$$
The Idea of the Proof

We wish to prove

\[ \langle \Psi \rangle = C_N^{-1} \text{Pf} \, U_P, \]

where

\[ U_P[j, k] := \langle P_j, P_k \rangle_R + \langle P_j, P_k \rangle_C. \]

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \right\} \]

\[ \times |\Delta(\alpha, \beta)| \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta), \]

First we need to expand \(|\Delta(\alpha, \beta)|\)
Vandermonde Determinants

\[ \Delta(\alpha, \beta) \text{ is given by} \]

\[
\det \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
\alpha_1 & \alpha_L & \beta_1 & \beta_1 & \beta_M \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\alpha_1^{N-1} & \alpha_L^{N-1} & \beta_1^{N-1} & \beta_1^{N-1} & \beta_M^{N-1}
\end{bmatrix}
\]
The Vandermonde Determinants

\( \Delta(\alpha, \beta) \) is given by

\[
\begin{vmatrix}
1 & 1 & ... & 1 & 1 & 1 & 1 & 1 \\
\alpha_1 & \alpha_L & ... & \beta_1 & \beta_1 & ... & \beta_M & \beta_M \\
\vdots & \vdots & ... & \vdots & \vdots & ... & \vdots & \vdots \\
\alpha_1^{N-1} & \alpha_L^{N-1} & ... & \beta_1^{N-1} & \beta_1^{N-1} & ... & \beta_M^{N-1} & \beta_M^{N-1}
\end{vmatrix}
\]

\[
= \left\{ \prod_{j<k} (\alpha_k - \alpha_j) \right\} \prod_{\ell=1}^{L} \prod_{m=1}^{M} |\beta_m - \alpha_\ell|^2 \\
\times \left\{ \prod_{m<n} |\beta_n - \beta_m|^2 |\beta_n - \overline{\beta_m}|^2 \right\} \prod_{m=1}^{M} 2i \Im(\beta_m).
\]
Vandermonde Determinants

Thus,

$$
|\det V^{\alpha, \beta}| = (-i)^m \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det V^{\alpha, \beta}.
$$
Vandermonde Determinants

Thus,

\[ |\det V_{\alpha,\beta}| = (-i)^M \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det V_{\alpha,\beta}. \]

And, \( \det V_{\alpha,\beta} \) is given by

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\alpha_1 & \cdots & \alpha_L & \beta_1 & \beta_1 & \cdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{N-1} & \cdots & \alpha_{N-1} & \beta_1^{N-1} & \beta_1^{N-1} & \cdots \\
\beta_1 & \cdots & \beta_M & \beta_M & \cdots & \beta_M \\
\end{vmatrix}
\]
Thus,

\[ \left| \det V^{\alpha,\beta} \right| = (-i)^M \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det V^{\alpha,\beta}. \]

And, \( \det V^{\alpha,\beta} \) is given by

\[
\begin{vmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
\alpha_1 & \alpha_L & \overline{\beta_1} & \beta_1 & \cdots & \beta_M \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{N-1} & \alpha_{N-1} & \overline{\beta_{N-1}} & \beta_{N-1} & \cdots & \beta_{M} \\
\end{vmatrix}
\]

Now, let \( P = \{ P_1, P_2, \ldots, P_N \} \) be a set of monic polynomials \( \deg P_n = n - 1 \).
Vandermonde Determinants

Thus,

\[ \left| \det V^{\alpha,\beta} \right| = (-i)^M \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det V^{\alpha,\beta}. \]

And, \( \det V^{\alpha,\beta} \) is given by

\[
\begin{vmatrix}
P_1(\alpha_1) & P_1(\alpha_L) & P_1(\beta_1) & P_1(\beta_1) & \cdots & P_1(\beta_M) \\
P_2(\alpha_1) & \cdots & P_2(\beta_1) & P_2(\beta_1) & \cdots & P_2(\beta_M) \\
\vdots & \cdots & \vdots & \ddots & \cdots & \vdots \\
P_N(\alpha_1) & \cdots & P_N(\beta_1) & P_N(\beta_1) & \cdots & P_N(\beta_M)
\end{vmatrix}
\]
Thus,
\[
\left| \det V^{\alpha,\beta} \right| = (-i)^M \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det V^{\alpha,\beta}.
\]

And, \( \det V^{\alpha,\beta} \) is given by
\[
\begin{vmatrix}
P_1(\alpha_1) & P_1(\alpha_L) & P_1(\bar{\beta_1}) & P_1(\beta_1) & P_1(\beta_M) \\
P_2(\alpha_1) & \cdots & P_2(\alpha_L) & P_2(\bar{\beta_1}) & P_2(\beta_1) & \cdots & P_2(\beta_M) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
P_N(\alpha_1) & \cdots & P_N(\alpha_L) & P_N(\bar{\beta_1}) & P_N(\beta_1) & \cdots & P_N(\beta_M)
\end{vmatrix}
\]

Call this matrix \( W^{\alpha,\beta} \).
Vandermonde Determinants

Thus,

\[ \left| \det V^{\alpha, \beta} \right| = (-i)^M \left\{ \prod_{j < k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det W^{\alpha, \beta}. \]
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$. 
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

$\mathcal{J}_{L}^{N} := \{\{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L)\}$.

To each $t \in \mathcal{J}_{L}^{N}$,

- there exists a unique $t' \in \mathcal{J}_{N-L}^{N}$ such that the images of $t$ and $t'$ are disjoint.
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

\[ \mathcal{J}^N_L := \{ \{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L) \}. \]

To each $t \in \mathcal{J}^N_L$,

- there exists a unique $t' \in \mathcal{J}^N_{N-L}$ such that the images of $t$ and $t'$ are disjoint.
- there is associated a canonical permutation $\nu_t \in S_N$. 
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha, \beta}$.

\[ \mathcal{J}_L^N := \left\{ \{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L) \right\}. \]

To each $t \in \mathcal{J}_L^N$,

- there exists a unique $t' \in \mathcal{J}_{N-L}^N$ such that the images of $t$ and $t'$ are disjoint.
- there is associated a canonical permutation $\nu_t \in S_N$.
- we define $\text{sgn}(t) := \text{sgn}(\nu_t)$. 
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

$$\mathcal{J}_L^N := \{\{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L)\}.$$  

Given $t, u \in \mathcal{J}_L^N$, we define $W_{t,u}^{\alpha,\beta}$ to be the $L \times L$ minor where

$$W_{t,u}^{\alpha,\beta}[j, k] := W^{\alpha,\beta}[t(j), u(k)].$$
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

\[ \mathcal{J}_L^N := \left\{ \{1, 2, \ldots, L\} \to \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L) \right\}. \]

Given $t, u \in \mathcal{J}_L^N$, we define $W_{t,u}^{\alpha,\beta}$ to be the $L \times L$ minor where

\[ W_{t,u}^{\alpha,\beta}[j, k] := W^{\alpha,\beta}[t(j), u(k)]. \]

The complimentary minor is given by $W_{t',u'}^{\alpha,\beta}$. 
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha, \beta}$.

\[ \mathcal{J}_L^N := \{ \{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L) \} \]

For fixed $u \in \mathcal{J}_L^N$, the Laplace expansion of $\det W^{\alpha, \beta}$ is given by

\[
\det W^{\alpha, \beta} = \text{sgn}(u) \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \det W_{t, u}^{\alpha, \beta} \cdot \det W_{t', u'}^{\alpha, \beta}.
\]
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

\[ \mathcal{J}_L^N := \left\{ \{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L) \right\}. \]

For fixed $u \in \mathcal{J}_L^N$, the Laplace expansion of $\det W^{\alpha,\beta}$ is given by

\[
\det W^{\alpha,\beta} = \text{sgn}(u) \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \det W_{t,u}^{\alpha,\beta} \cdot \det W_{t',u'}^{\alpha,\beta}.
\]

Let $i \in \mathcal{J}_L^N$ be the identity map. Then, for every $t \in \mathcal{J}_L^N$, the entries of $W_{t,i}^{\alpha,\beta}$ do not depend on $\beta$. 
The Laplace Expansion of the Determinant

We wish to enumerate the minors of $W^{\alpha,\beta}$.

$\mathcal{J}_L^N := \{\{1, 2, \ldots, L\} \to \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L)\}$.

For fixed $u \in \mathcal{J}_L^N$, the Laplace expansion of $\det W^{\alpha,\beta}$ is given by

$$\det W^{\alpha,\beta} = \operatorname{sgn}(u) \sum_{t \in \mathcal{J}_L^N} \operatorname{sgn}(t) \det W_{t,u}^{\alpha,\beta} \cdot \det W_{t',u'}^{\alpha,\beta}.$$

Let $i \in \mathcal{J}_L^N$ be the identity map. Then, for every $t \in \mathcal{J}_L^N$, the entries of $W_{t,i}^{\alpha,\beta}$ do not depend on $\beta$. Similarly, the entries of $W_{t',i'}^{\alpha,\beta}$ do not depend on $\alpha$. 

We wish to enumerate the minors of $W^{\alpha,\beta}$.

$\mathcal{J}^N_L := \{\{1, 2, \ldots, L\} \rightarrow \{1, 2, \ldots, N\} : t(1) < t(2) < \cdots < t(L)\}$. 

Thus,

$$
\det W^{\alpha,\beta} = \sum_{t \in \mathcal{J}^N_L} \text{sgn}(t) \det W^\alpha_{t,i} \cdot \det W^\beta_{t',i'}.
$$
We have,

\[ |\Delta(\alpha, \beta)| = (-i)^M \left\{ \prod_{j < k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det W^{\alpha, \beta}. \]
Vandermonde Determinants

We have,

\[ |\Delta(\alpha, \beta)| = (-i)^M \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \det W^{\alpha, \beta}. \]
Vandermonde Determinants

We have,

\[ |\Delta(\alpha, \beta)| = (-i)^M \left\{ \prod_{j < k} \text{sgn}(\alpha_k - \alpha_j) \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \]

\[ \times \sum_{t \in J_L^N} \text{sgn}(t) \det W_{t,i}^\alpha \cdot \det W_{t',i'}^\beta. \]
Vandermonde Determinants

We have,

$$|\Delta(\alpha, \beta)| = \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \left\{ \det W_{t, i}^\alpha \prod_{j < k} \text{sgn}(\alpha_k - \alpha_j) \right\}$$

$$\times \left\{ \det W_{t', i'} (-i)^M \prod_{m=1}^M \text{sgn} \mathcal{S}(\beta_m) \right\}.$$
The Idea of the Proof

\[ \langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\bar{\beta}_m) \right\} \times |\Delta(\alpha, \beta)| d\lambda_L(\alpha) d\lambda_{2M}(\beta), \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \frac{1}{L! M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta}_m) \right\} \times |\Delta(\alpha, \beta)| \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta), \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^L \varphi(\alpha_\ell) \prod_{m=1}^M \varphi(\beta_m) \varphi(\overline{\beta}_m) \right\} \]

\[ \times \left( \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \left\{ \det W_{t,i}^\alpha \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} \right. \]

\[ \times \left\{ \det W_{t',i'}^\beta (-i)^M \prod_{m=1}^M \text{sgn} \Im(\beta_m) \right\} \left. \right\} \text{d}\lambda_L(\alpha) \text{d}\lambda_{2M}(\beta), \]
The Idea of the Proof

\[ \langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \frac{1}{L!M!} \int_{\mathbb{R}^L \times \mathbb{C}^M} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_{\ell}) \prod_{m=1}^{M} \varphi(\beta_m)\varphi(\overline{\beta_m}) \right\} \]

\times \left( \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \right) \left\{ \det W_{t,i}^\alpha \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} \]

\times \left\{ \det W_{t',i'}^{\beta} (-i)^M \prod_{m=1}^{M} \text{sgn} \Im(\beta_m) \right\} \right) \, d\lambda_L(\alpha) \, d\lambda_{2M}(\beta), \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{F}^N_L} \text{sgn}(t) \frac{1}{L!M!} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \]

\times \left\{ \det W_{t,i}^{\alpha} \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\}

\times \left\{ \det W_{t',i'}^{\beta} (-i)^M \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta}_m) \text{sgn} \Im(\beta_m) \right\}

\times d\lambda_L(\alpha) d\lambda_{2M}(\beta),
The Idea of the Proof

\[
\langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \frac{1}{L!M!} \int_{\mathbb{R}^L} \int_{\mathbb{C}^M} \\
\times \left\{ \det W^{\alpha}_{t,i} \prod_{\ell=1}^L \varphi(\alpha_\ell) \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} \\
\times \left\{ \det W^{\beta}_{t',i'} (-i)^M \prod_{m=1}^M \varphi(\beta_m) \varphi(\overline{\beta}_m) \text{sgn} \mathcal{S}(\beta_m) \right\} \\
\times d\lambda_L(\alpha) \, d\lambda_{2M}(\beta),
\]
The Idea of the Proof

\[ \langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \]

\[ \times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{t,i}^\alpha \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_{\ell}) \right\} \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} d\lambda_L(\alpha) \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \]

\[ \times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_t^{\alpha} \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \left\{ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) \right\} d\lambda_L(\alpha) \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'}^{\beta} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]

It is well known (De Bruijn 1955), that

\[ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) = \text{Pf} T^{\alpha} \quad \text{where} \quad T^{\alpha}[j, k] = \text{sgn}(\alpha_k - \alpha_j). \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{T}_L^N} \text{sgn}(t) \]

\[ \times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{t,i}^\alpha \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_{\ell}) \right\} \text{Pf } T^\alpha d\lambda_L(\alpha) \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'}^\beta \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn } \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]

It is well known (De Bruijn 1955), that

\[ \prod_{j<k} \text{sgn}(\alpha_k - \alpha_j) = \text{Pf } T^\alpha \text{ where } T^\alpha[j, k] = \text{sgn}(\alpha_k - \alpha_j). \]
The Idea of the Proof

\[ \langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \]

\[ \times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{t,i}^\alpha \left\{ \prod_{\ell=1}^L \varphi(\alpha_\ell) \right\} \text{Pf} T^\alpha d\lambda_L(\alpha) \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'}^\beta \left\{ \prod_{m=1}^M \varphi(\beta_m) \varphi(\beta_m^*) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \]

\[ \times \frac{1}{L!} \int_{\mathbb{R}^L} \det W_{t,i}^\alpha \left\{ \prod_{\ell=1}^{L} \varphi(\alpha_\ell) \right\} \text{Pf} T^\alpha d\lambda_L(\alpha) \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'}^\beta \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]

\[ * = \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(t_\tau)(2\ell-1)}, P_{(t_\tau)(2\ell)} \rangle_{\mathbb{R}} \right\}. \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{T}_L^N} \text{sgn}(t) \]

\[ \times \left\{ \frac{1}{2^LL!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^L \langle P_{(t \circ \tau)(2\ell-1)}, P_{(t \circ \tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right\} \]

\[ \times \left( \frac{-i}{M} \right)^M \int_{\mathbb{C}^M} \det W_{t', i'} \left\{ \prod_{m=1}^M \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]
The Idea of the Proof

\[ \langle \Psi \rangle = \mathcal{C}_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{F}_L^N} \text{sgn}(t) \times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(t \circ \tau)}(2\ell-1), P_{(t \circ \tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right\} \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{\nu, i'} \left\{ \prod_{m=1}^{M} \varphi(\beta_m) \varphi(\overline{\beta_m}) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{H}_N^L} \text{sgn}(t) \]

\[ \times \left\{ \frac{1}{2^L L!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^L \langle P(t \circ \tau)(2\ell-1), P(t \circ \tau)(2\ell) \rangle_{\mathbb{R}} \right\} \right\} \]

\[ \times \frac{(-i)^M}{M!} \int_{\mathbb{C}^M} \det W_{t',i'}^\beta \left\{ \prod_{m=1}^M \varphi(\beta_m) \varphi(\beta_m) \text{sgn} \Im(\beta_m) \right\} d\lambda_{2M}(\beta). \]

\[ \ast = \frac{1}{2^M M!} \sum_{\sigma \in S_{2M}} \text{sgn}(\sigma) \prod_{m=1}^M \langle P(t' \circ \sigma)(2m-1), P(t' \circ \sigma)(2m) \rangle_{\mathbb{C}}. \]
The Idea of the Proof

\[ \langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{T}_L^N} \text{sgn}(t) \]
\[ \times \left\{ \frac{1}{2L L!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(t \circ \tau)(2\ell-1)}, P_{(t \circ \tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right\} \]
\[ \times \left\{ \frac{1}{2M M!} \sum_{\sigma \in S_{2M}} \text{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(t' \circ \sigma)(2m-1)}, P_{(t' \circ \sigma)(2m)} \rangle_{\mathbb{C}} \right\}. \]
The Idea of the Proof

\[ \langle \Psi \rangle = c_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \]

\[ \times \left\{ \frac{1}{2L L!} \sum_{\tau \in S_{2L}} \text{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P(t_{\circ \tau})_{(2\ell-1)}, P(t_{\circ \tau})_{(2\ell)} \rangle_{\mathbb{R}} \right\} \right\} \]

\[ \times \left\{ \frac{1}{2M M!} \sum_{\sigma \in S_{2M}} \text{sgn}(\sigma) \prod_{m=1}^{M} \langle P(t'_{\circ \sigma})_{(2m-1)}, P(t'_{\circ \sigma})_{(2m)} \rangle_{\mathbb{C}} \right\}. \]

Define the \( N \times N \) matrices \( R \) and \( C \) by

\[ R[j, k] = \langle P_j, P_k \rangle_{\mathbb{R}} \quad \text{and} \quad C[j, k] = \langle P_j, P_k \rangle_{\mathbb{C}}. \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{J}_N^L} \operatorname{sgn}(t) \]

\times \left\{ \frac{1}{2L L!} \sum_{\tau \in S_{2L}} \operatorname{sgn}(\tau) \left\{ \prod_{\ell=1}^{L} \langle P_{(t o \tau)(2\ell - 1)}, P_{(t o \tau)(2\ell)} \rangle_{\mathbb{R}} \right\} \right\}

\times \left\{ \frac{1}{2M M!} \sum_{\sigma \in S_{2M}} \operatorname{sgn}(\sigma) \prod_{m=1}^{M} \langle P_{(t' o \sigma)(2m - 1)}, P_{(t' o \sigma)(2m)} \rangle_{\mathbb{C}} \right\}.

These are (resp.) the Pfaffians of the minors \( R_{t,t} \) and \( C_{t',t'} \).
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{I}_L^N} \text{sgn}(t) \text{Pf } R_{t,t} \cdot \text{Pf } C_{t',t'} \, . \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \sum_{(L,M)} \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \text{Pf} \, R_{t,t} \cdot \text{Pf} \, C'_{t',t'} \]

It can be verified that

\[ \sum_{(L,M)} \sum_{t \in \mathcal{J}_L^N} \text{sgn}(t) \text{Pf} \, R_{t,t} \cdot \text{Pf} \, C'_{t',t'} = \text{Pf} (R + C) \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \text{Pf}(R + C). \]
The Idea of the Proof

\[ \langle \Psi \rangle = C_N^{-1} \text{Pf}(R + C). \]

\[ R + C = U_P. \]
The Idea of the Proof

Finally,

$$\langle \Psi \rangle = C_N^{-1} \text{Pf}(U_P).$$
Theorem 9 (S- 2006). Let $N = 2J$, and let

$$\mathbf{P} = \{P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma)\}$$

be a set of monic polynomials with $\deg P_n = n - 1$. Then,

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \text{Pf} \ U_\mathbf{P},$$

the Pfaffian of $U_\mathbf{P}$, where $U_\mathbf{P}$ is the $N \times N$ matrix where,

$$U_\mathbf{P}[j, k] := \langle P_j, P_k \rangle_\mathbb{R} + \langle P_j, P_k \rangle_\mathbb{C},$$

where $\langle P, Q \rangle_\mathbb{R}$ and $\langle P, Q \rangle_\mathbb{C}$ are defined with respect to

$$\varphi(\gamma) = \psi(\gamma) e^{-\gamma^2/2} \left\{ \text{erfc}(\sqrt{2}|\text{Im}(\gamma)|) \right\}^{1/2}.$$
Theorem 10 (S- 2005). Let \( N = 2J \), and let

\[
P = \{ P_1(\gamma), P_2(\gamma), \ldots, P_N(\gamma) \}
\]

be a set of monic polynomials with \( \deg P_n = n - 1 \). Then,

\[
F_N(\Phi; s) = \text{Pf } U_P,
\]

the Pfaffian of \( U_P \), where \( U_P \) is the \( N \times N \) matrix where,

\[
U_P[j, k] := \langle P_j, P_k \rangle_R + \langle P_j, P_k \rangle_C,
\]

where \( \langle P, Q \rangle_R \) and \( \langle P, Q \rangle_C \) are defined with respect to

\[
\varphi(\gamma) = \phi(\gamma)^{-s}.
\]