Metallic quantum ferromagnets

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An overview of quantum phase transitions (QPTs) in metallic ferromagnets, discussing both experimental and theoretical aspects, is given. These QPTs can be classified with respect to the presence and strength of quenched disorder: Clean systems generically show a discontinuous, or first-order, QPT from a ferromagnetic to a paramagnetic state as a function of some control parameter, as predicted by theory. Disordered systems are much more complicated, depending on the disorder strength and the distance from the QPT. In many disordered materials the QPT is continuous, or second order, and Griffiths-phase effects coexist with QPT singularities near the transition. In other systems the transition from the ferromagnetic state at low temperatures is to a different type of long-range order, such as an antiferromagnetic or a spin-density-wave state. In still other materials a transition to a state with glasslike spin dynamics is suspected. The review provides a comprehensive discussion of the current understanding of these various transitions and of the relation between experiment and theory.

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I. INTRODUCTION

Metallic ferromagnets have been studied since ancient times, as this class of materials includes elemental iron, which gave ferromagnetism its name. Detailed studies in the early 1900s led to one of the first examples of mean-field theory (Weiss, 1907). A more elaborate mean-field theory by Stoner (1938) explained how a nonzero magnetization can arise from a spontaneous splitting of the conduction band. When it became clear, 30 years later, that mean-field theory does not correctly describe the behavior close to the phase transition, ferromagnetism became one of the testing grounds for the theory of critical phenomena (Stanley, 1971; Wilson and Kogut, 1974). More recently metallic ferromagnets with low Curie temperatures, ranging from tens of degrees to a few degrees, or even lower, have attracted much attention. Many of these materials allow for decreasing the Curie temperature even further, by applying pressure or by changing the chemical composition. This makes possible the study of the quantum phase transition (QPT) that occurs at zero temperature and for fundamental reasons must be quite different in nature from the thermal phase transition observed at a nonzero Curie temperature. Over the years it again became clear that the quantum version of mean-field theory does not correctly describe the behavior close to the transition, contrary to early suggestions.

This review summarizes the experimental and theoretical understanding of this quantum phase transition. Our discussion of experimental results is restricted to materials where an instability of a ferromagnetic phase at low temperatures is clearly observed and reasonably well characterized. In parallel to this discussion we describe the relevant theoretical ideas and the extent to which they explain, and in some cases predict, the experimental observations. In this section we start
with some general remarks about quantum phase transitions and then turn to the one in metallic ferromagnets.

A. General remarks on quantum phase transitions

QPTs have been discussed for many years and remain a subject of great interest (Hertz, 1976; Sachdev, 1999). Whereas classical or thermal phase transitions occur at a nonzero transition temperature and are driven by thermal fluctuations, QPTs occur at zero temperature, $T = 0$, as a function of some nonthermal control parameter (typical examples are pressure, composition, or an external magnetic field) and are driven by quantum fluctuations. The ways in which the description of QPTs differs from that of their classical counterparts are subtle and took a long time to understand. Early on it was realized that at a mean-field level the description is the same for both quantum and classical phase transitions. Indeed, the earliest theory of a QPT was the Stoner theory of ferromagnetism (Stoner, 1938). Stoner considered the case of itinerant ferromagnets, where the conduction electrons are responsible for the ferromagnetism, and developed a mean-field theory that describes both the classical and the quantum ferromagnetic transition.

Important mathematical developments were the Trotter formula (Trotter, 1959), and the coherent-state formalism (Casher, Lurié, and Revzen, 1968), which proved useful for representing the partition function of quantum spin systems in terms of a functional integral (Suzuki, 1976a, 1976b). It implied, at least for certain spin models, that a quantum phase transition in a system with $d$ spatial dimensions could be described in terms of the corresponding classical phase transition in an effective dimension $d_{\text{eff}} = d + 1$. An example is the Ising model in a transverse field (DeGennes, 1963; Stinchcombe, 1973). The crucial observation was that the functional-integral representation of the partition function contains an integration over an auxiliary variable (usually referred to as imaginary time) that extends from zero to the inverse temperature $1/T$. At $T = 0$, this integration range becomes infinite and mimics an additional spatial integration in the thermodynamic limit. If space and time scale in the same way, then $d_{\text{eff}} = d + 1$ follows. For the classical Ising model, in particular, the upper critical dimension, above which mean-field theory provides an exact description of the transition, is $d_{\text{c}} = 4$. It follows that the critical behavior of the quantum Ising model in a transverse field in $d > 3$ is mean field like (Suzuki, 1976a). More generally, the statics and the dynamics are intrinsically coupled at QPTs, unlike the case of classical phase transitions, where the dynamic critical phenomena are decoupled from the statics (Ferrell et al., 1967, 1968; Halperin and Hohenberg, 1967, 1969; Hohenberg and Halperin, 1977).

This leads to the following general point: For classical critical phenomena, the dynamic universality classes are much smaller (and therefore more numerous) than the static ones. Physically, this is due to the fact that the order-parameter fluctuations that determine the universality class can be conserved (such as in, e.g., a ferromagnet) or nonconserved (such as in, e.g., an antiferromagnet), and they can couple to any number of other slow or soft modes or excitations, with each of these cases realizing a different universality class (Hohenberg and Halperin, 1977). By the same argument one expects quantum phase transitions in metals to be different from those in insulators because the respective dynamical processes are very different.

Hertz (1976), in an important paper, among other things, generalized the Trotter-Suzuki formulation to the case where space and time do not scale the same way. He showed that if the slow order-parameter time scale $t_\xi$ at a continuous QPT diverges as $t_\xi \propto \xi^z$, with $\xi$ the correlation length and $z$ the dynamical scaling exponent (which in general is not equal to unity), then the imaginary-time integral is analogous to a spatial integration over an additional $z$ spatial dimensions. For such a class of problems the critical behavior at the continuous QPT is equivalent to that at the corresponding classical transition in $d_{\text{eff}} = d + z$ dimensions. At this point it seemed that QPTs were, in fact, not fundamentally different from their classical counterparts. The statics and the dynamics couple, leading to an effective dimension different from the physical spatial dimension, and the number of universality classes is different, but the technical machinery that had been developed to solve the classical phase-transition problem (Wilson and Kogut, 1974; Ma, 1976; Fisher, 1983) could be generalized to treat QPTs as well and map them onto classical transitions in a different dimension.

These considerations assume that the phase transition separates an ordered phase from a disordered one, with the ordered phase characterized by a local order parameter. For the ferromagnetic transition that is the subject of this review, this is indeed the case. It should be mentioned, however, that there are very interesting phase transitions, both classical and quantum, that do not allow for a description in terms of a local order parameter. One example is provided by spin liquids (Balents, 2010), others by the quantum Hall effects (von Klitzing, Dorda, and Pepper, 1980; Tsui, Stormer, and Gossard, 1982) and topological insulators (Hasan and Kane, 2010; Qi and Zhang, 2011). Other interesting cases are the Anderson and Anderson-Mott metal-insulator...
transitions (see footnote 4). It was proposed that for these transitions, and indeed for all QPTs, the von Neumann entanglement entropy $S_c$ is a useful concept since it displays nonanalyticities characteristic of the QPT (Kopp, Jia, and Chakravarty, 2007). $S_c$ tends to scale with the area of the subsystem rather than its volume and provides interesting connections between correlated electrons, quantum information theory, and the thermodynamics of black holes (Eisert, Cramer, and Plenio, 2010).

B. Quantum ferromagnetic transitions in metals

Hertz’s prime example was the same as Stoner’s, namely, an itinerant ferromagnet. Here the magnetization serves as an order parameter, and Hertz (1976) derived a dynamical Landau-Ginzburg-Wilson (LGW) functional for this transition by considering a model of itinerant electrons that interact only through a contact potential in the particle-hole spin-triplet channel. He analyzed this LGW functional by means of renormalization-group (RG) methods. He concluded that in this case the dynamical critical exponent has the value $z = 3$, and that the QPT for an itinerant Heisenberg ferromagnet hence maps onto the corresponding classical transition in $d_{\text{eff}} = d + 3$ dimensions. Since the upper critical dimension for classical ferromagnets is $d^c = 4$, this seemed to imply that Stoner theory for the critical behavior was exact in the physical spatial dimensions $d = 2$ and $3$. This in turn implied that the transition was generically continuous or second order, with mean-field static critical exponents. Preceding Hertz’s work, Moriya and collaborators in the early 1970s had developed a description of itinerant quantum ferromagnets that one would now classify as a self-consistent one-loop theory [historically, it was often referred to as self-consistently renormalized (SCR) or spin-fluctuation theory]; this work was summarized by Moriya (1985).Millis (1993) used Hertz’s RG framework to study the behavior at low but nonzero temperature and the crossover between the quantum and classical scaling behaviors. Most of the explicit results confirmed the earlier results of the spin-fluctuation theory. This combined body of work is often referred to as Hertz-Millis-Moriya or Hertz-Millis theory. We will discuss its basic features and results in Sec. III.C.2.

A related but separate line of investigations dealt with quantitative issues regarding the strength of the magnetism, and the properties of the ordered phase, in itinerant ferromagnets. It was realized early on that Stoner theory and its extension to finite temperature (Edwards and Wohlforth, 1968) leaves key questions unanswered, especially for metals with low Curie temperatures $T_C$: First, why is the exchange energy, which can be extracted from band-structure probes or from careful analysis of the magnetic equation of state, typically at least an order of magnitude larger than $k_B T_C$? If the order was destroyed solely by a thermal smearing of the Fermi function, the two would be expected to be of similar magnitude. Second, why is the ordered moment in the low-temperature limit only a small fraction of the fluctuating moment as extracted from the Curie constant in the temperature dependent susceptibility? Third, why is the temperature dependence of the magnetization at low temperature proportional to $T^2$ rather than $T^{3/2}$, as would be expected from including spin-wave excitations?

The key to answering these questions, and to achieving a quantitative description of itinerant ferromagnets with low ordering temperatures, was to include the effect of fluctuations of the local magnetization, as demonstrated by Murata and Doniach (1972). More comprehensive models were developed in the 1970s by Moriya and collaborators (Moriya, 1985) in the spin-fluctuation-theory work already mentioned. As inelastic neutron scattering became feasible, which demonstrated the existence of magnetic fluctuations and allowed for their quantitative parametrization (Ishikawa et al., 1982; Bernhoeft et al., 1983), it became possible to accurately model key material properties such as $T_C$, the low-temperature ordered moment, and its temperature dependence, as well as the temperature dependence of the magnetic susceptibility and the associated fluctuating moment, in a further development of the SCR spin-fluctuation approach (Lonzarich and Taillefer, 1985).

Returning to the statistical-mechanics description of the phase transition itself, a key result of both the SCR theories and Hertz’s RG description of clean metallic ferromagnets was the value of the dynamical exponent $z = 3$. This can be made plausible independent of the technical details of Hertz’s theory, and, more importantly, independent of whether or not the conduction electrons themselves are responsible for the magnetism. In the absence of soft modes other than the order-parameter fluctuations, the bare order-parameter susceptibility $\chi_{\text{OP}}$ at criticality as a function of the frequency $\omega$ and the wave number $k$ has the form (Hohenberg and Halperin, 1977)

$$\chi_{\text{OP}}^{-1}(k, \omega) = -i\omega/\gamma + k^2,$$  \hspace{1cm} (1.1a)

if the order parameter is not a conserved quantity, or

$$\chi_{\text{OP}}^{\text{rl}}(k, \omega) = -i\omega/\lambda k^2 + k^2,$$  \hspace{1cm} (1.1b)

if it is, with $\gamma$ and $\lambda$ kinetic coefficients. At $T > 0$, or at $T = 0$ in the presence of quenched disorder, $\gamma$ and $\lambda$ are weakly $k$ dependent and approach constants as $k \rightarrow 0$. However, in clean systems at $T = 0$ these coefficients do not exist in the limit of zero frequency and wave number, and in metallic systems their effective behavior is $\gamma \propto \lambda \propto 1/k$. For a nonconserved order parameter this leads to $z = 1$, as in the case of a quantum antiferromagnet (Chakravarty, Halperin, and Nelson, 1989), or an Ising model in a transverse field (Suzuki, 1976a). For a ferromagnet, where the order parameter is conserved, we find from Eq. (1.1b) $z = 3$ in the clean case, and $z = 4$ in the disordered case. This is consistent with Hertz’s explicit calculation for a specific model. Equations (1.1) do not get qualitatively changed by renormalizations, provided $d_{\text{eff}} = d + z$ is greater than the upper critical dimension: The coupling between the statics and the dynamics ensures that the critical exponent $\eta$ is zero and the exponents in Eqs. (1.1) remain unchanged. Simple mean-field arguments, including Eqs. (1.1), are therefore self-consistently valid for all $d > d^c - z$, the static critical behavior is mean field like, and the dynamical critical exponent is the one that follows from Eqs. (1.1). However, all of these considerations are valid only under the assumption that there are no other soft modes that couple to the order parameter. In metallic

5For a definition of critical exponents, see Appendix B.
ferromagnets this assumption is not valid as explained in detail in Sec. III.

The experimental situation through the 1990s was confusing: In some materials a second-order transition was observed, but many others showed a first-order transition. Within mean-field theory, the standard explanation for a first-order transition is that the coefficient of the quartic term in the Landau field theory, the standard explanation for a first-order transition. The physical reason underlying this universal conclusion is a coupling of the magnetization to electronic soft modes that exist in any metal, which leads to a fluctuation-induced first-order transition. The same conclusion was reached by other groups (Chubukov, Pépin, and Rech, 2004; Maslov, Chubukov, and Saha, 2006; Rech, Pépin, and Chubukov, 2006). This theoretical work was later generalized to include the effects of an external magnetic field, which leads to tricritical wings in the phase diagram (Belitz, Kirkpatrick, and Rollbühler, 2005). Since the role of the electronic soft modes diminishes with increasing temperature, this theory predicts that in clean systems there necessarily exists a tricritical point in the phase diagram, i.e., a temperature that separates a line of first-order transitions at low temperatures from a line of second-order transitions at higher temperatures as the control parameter is varied. In addition, BKV showed that nonmagnetic quenched disorder suppresses the tricritical temperature, and that the transition remains second order down to zero temperature if the disorder strength exceeds a critical value.

An important generalization of the theory was the realization that it is equally valid for localized-moment ferromagnets as for itinerant ones (Kirkpatrick and Belitz, 2012a); the previous focus on itinerant magnets was historically rooted as for itinerant ones (Kirkpatrick and Belitz, 2012a); the previous focus on itinerant magnets was historically rooted. A much more general mechanism for a first-order transition was proposed in 1999, when two of the present authors, together with Thomas Vojta, showed theoretically that the QPT in two-dimensional and three-dimensional metallic systems from a paramagnetic (PM) phase to a homogeneous ferromagnetic one is generically first order, provided the material is sufficiently clean (Belitz, Kirkpatrick, and Vojta, 1999, referred to as BKV). The physical reason underlying this universal conclusion is a coupling of the magnetization to electronic soft modes that exist in any metal, which leads to a fluctuation-induced first-order transition. The same conclusion was reached by other groups (Chubukov, Pépin, and Rech, 2004; Maslov, Chubukov, and Saha, 2006; Rech, Pépin, and Chubukov, 2006). This theoretical work was later generalized to include the effects of an external magnetic field, which leads to tricritical wings in the phase diagram (Belitz, Kirkpatrick, and Rollbühler, 2005). Since the role of the electronic soft modes diminishes with increasing temperature, this theory predicts that in clean systems there necessarily exists a tricritical point in the phase diagram, i.e., a temperature that separates a line of first-order transitions at low temperatures from a line of second-order transitions at higher temperatures as the control parameter is varied. In addition, BKV showed that nonmagnetic quenched disorder suppresses the tricritical temperature, and that the transition remains second order down to zero temperature if the disorder strength exceeds a critical value.

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Many experiments are consistent with these predictions, and over time experiments on cleaner samples, or at lower temperatures, or both, showed a first-order transition even in cases where previously a continuous transition had been found. The predicted tricritical point and associated tricritical wings have also been observed in many systems. For a representative example of this type of phase diagram, showing a tricritical point (TCP), the tricritical wings, and “quantum critical end points” (QCEPs) at the wing tips, see Fig. 1.

![FIG. 1. Observed phase diagram of UGe₂ in the space spanned by temperature (T), pressure (P), and magnetic field (H). Solid red curves represent lines of second-order transitions, and blue planes represent first-order transitions. Also shown are the tricritical point (TCP), and the extrapolated “quantum-critical end points” (QCEP) at the wing tips. From Kotegawa, Taufour et al., 2011.](image)

Strongly disordered materials, on the other hand, almost always show a continuous transition, also in agreement with the theoretical prediction. There are, however, exceptions from these general patterns, which are discussed in Sec. II.C.1.

These predictions and observations are for systems where the transition is to a homogeneous ferromagnetic state; the schematic phase diagrams for the discontinuous and continuous cases, respectively, are shown in Figs. 2(a) and 2(b). In other materials, magnetic order of a different kind is found to compete with homogeneous ferromagnetism at low temperatures, as schematically illustrated in Fig. 2(c). In strongly disordered systems, spin-glass freezing and quantum Griffiths effects may occur at low temperatures and augment or compete with critical behavior; see Fig. 2(d). These effects are discussed in detail in Secs. II.D, II.E, III.D, and III.E.

The striking difference between the predictions of BKV and Hertz theory is due to a coupling of the order-parameter fluctuations to electronic degrees of freedom. Hertz theory treats this coupling in too simple an approximation to capture all of its qualitative effects. In metals at T = 0 there are soft or gapless two-fermion excitations that couple to the magnetic order-parameter fluctuations in important ways. In effect, the combined fermionic and bosonic (order-parameter) fluctuations determine the quantum universality class in all spatial dimensions d < 3. As a result of this coupling, the upper critical dimension is d_c^Z = 3, rather than d_c^Z = 1 as predicted by Hertz theory, and the transition is first order, rather than continuous with mean-field exponents. The mechanism behind this phenomenon is similar to what is known as a fluctuation-induced first-order transition in classical phase transitions (Halperin, Lubensky, and Ma, 1974; Chen, Lubensky, and Nelson, 1978), but it is different in at least one crucial way; cf. Secs. III.B.2 and IV.A. Two well-known classical examples of a fluctuation-induced first-order transition are the superconducting Bardeen-Cooper-Schrieffer...
The continuous quantum ferromagnetic transition in disordered metals, in systems where the disorder is strong enough to suppress the tricritical temperature to zero, has also been studied in detail theoretically (Kirkpatrick and Belitz, 1996; Belitz et al., 2001a, 2001b). In this case the itinerant electrons are moving diffusively, rather than ballistically. Because this is a slower process, there is an effective enhancement of the exchange interaction that causes ferromagnetism, and some crucial signs are changed compared to the clean case. The net result is that the second-order transition predicted by Hertz theory becomes, so to speak, even more continuous by the coupling to the electronic soft modes: For example, the theory predicts that in $d = 3$ the critical exponent ($\beta$) is equal to 2, compared to $\beta = 1/2$ in Hertz theory. This large value of $\beta$ may give the impression of a “smeared transition,” even though there still is a sharp critical point. This, as well as the predicted values of other exponents, is consistent with numerous experiments in disordered systems, as we will discuss. In related developments, much work has been done on Griffiths singularities and Griffiths phases in disordered metallic magnets. Depending on the nature and symmetry of the order parameter, these theories predict that in some systems the Griffiths-phase effects are very weak, while in others they lead to strong power-law singularities with continuously varying exponents, and in yet others they completely destroy the sharp quantum phase transition [for a review, see Vojta (2010)]. If these effects are important, they will be superimposed on the critical behavior.

Finally, there are theories that suggest that in some metallic systems an inhomogeneous magnetic phase may form in between the paramagnetic and the homogeneous ferromagnetic state at low $T$. This was first suggested by Belitz, Kirkpatrick, and Vojta (1997), and has been explored in detail by others. Spiral phases, spin nematics, and spin-density waves have been proposed to appear between the uniform ferromagnet and the paramagnetic phase (Chubukov, Pépin, and Rech, 2004; Maslov, Chubukov, and Saha, 2006; Rech, Pépin, and Chubukov, 2006; Efremov, Betouras, and Chubukov, 2008; Chubukov and Maslov, 2009; Conduit, Green, and Simons, 2009; Karahasanovic, Krüger, and Green, 2012). We discuss these and related theories in Sec. III.E.

II. EXPERIMENTAL RESULTS

Here we discuss experimental results organized with respect to the observed phase diagrams as shown in Fig. 2.

A. General remarks

During the last two decades a large number of FM metals have been found that (1) have a low Curie temperature, and (2) can be driven across a ferromagnet-to-paramagnet QPT. The control parameter is often either hydrostatic pressure or uniaxial stress, but the transition can also be triggered by composition, or an external magnetic field. The initial motivation was to look for a ferromagnetic quantum-critical

\[ T \]
point (QCP), and possibly novel states of matter in its vicinity, as had been found in many antiferromagnetic (AFM) metals (Grosche et al., 1996; Mathur et al., 1998; Park et al., 2006; von Löhnneysen et al., 2007; Gegenwart, Si, and Steglich, 2008). It soon became clear, however, that the FM case is quite different from the AFM one. Instead of displaying a quantum-critical point, many systems were found to undergo a first-order QPT, with a tricritical point in the phase diagram separating a line of second-order transitions at relatively high temperatures from a line of first-order transitions at low temperatures. In several of these materials the existence of a tricritical point was confirmed by the observation of tricritical wings upon the application of an external magnetic field $H$, as shown schematically in Fig. 2(a). Some systems, such as ZrZn$_2$, were initially reported to have a QCP, but with increasing sample quality the transition at low temperatures was found to be first order. The first-order transition occurs in a large variety of materials, including 3d transition metals as well as 4f- and 5f-electron systems; see Tables I and II. Some systems do show a continuous QPT to the lowest temperatures observed; see Tables III, IV, and V and Fig. 2(b). Several of these are either strongly disordered, as judged by their residual resistivities, or their crystal structure makes them quasi one dimensional. Finally, the expectation of additional phases was borne out. In some systems the long-range order changes from ferromagnetic to modulated spin-density-wave (SDW) or AFM order [see Fig. 2(c)], and strongly disordered systems often show a spin-glass-like phase in the tail of the phase diagram; see Fig. 2(d). Accordingly, we distinguish four categories of metallic quantum ferromagnets, namely, (1) systems that display a first-order QPT; (2) systems that display, or are suspected to display, a QCP; (3) systems that undergo a phase transition to a different type of magnetic order before the FM quantum phase transition is reached; and (4) systems with spin-glass-like characteristics or other manifestations of strong disorder at low temperatures. This phenomenological classification, which is independent of the microscopical origin of the magnetism, is reflected in Fig. 2 and Tables I–VII. For each of these categories we discuss a number of representative materials in which the QPT has been reasonably well characterized. This list of materials is not exhaustive.

We also mention that superconductivity has been found to coexist with itinerant ferromagnetism in four U-based FM metals: UGe$_2$ (Saxena et al., 2000), URhGe (Aoki et al., 2001; Yelland et al., 2011), UCoGe (Huy, Gasparini, Klaasse et al., 2007), and UIr (Kobayashi et al., 2006). While very interesting, this topic is outside the scope of this review and will be mentioned only in passing. For a related review, see Pfleiderer (2009). Another interesting class of materials that we do not cover are ferromagnetic semiconductors which have recently been reviewed by Jungwirth et al. (2006).

B. Systems showing a discontinuous transition

We first discuss systems in which there is strong evidence for a first-order transition at low temperatures. These include the transition-metal compounds MnSi and ZrZn$_2$, several uranium-based compounds, and some other materials; their properties are summarized in Tables I and II. The widespread pattern of first-order transitions near the QPT is consistent with fundamental arguments such as the BKV theory (Belitz, Kirkpatrick, and Vojta, 1999; Belitz, Kirkpatrick, and Rollbühler, 2005), which for clean ferromagnets predicted a first-order quantum phase transition at $T = 0$, a tricritical point in the phase diagram, and associated tricritical wings in an external magnetic field. This theory is reviewed in Sec. III, where we give a detailed discussion of the relation between theory and experiment.

1. Transition-metal compounds

a. MnSi

MnSi is a very well-studied material in which the search for a FM QCP resulted in the observation of a first-order quantum phase transition. The transition temperature at ambient pressure is $T_C \approx 29.5$ K, and the application of hydrostatic pressure suppresses $T_C$ to zero at a critical pressure $p_c \approx 14.6$ kbar (Pfleiderer, McMullan, and Lonzarich, 1994; Pfleiderer et al., 1997). This compound is actually a weak helimagnet (Ishikawa et al., 1976) with a complicated phase diagram [see Mühlbauer et al. (2009) and references therein]. However, the long wavelength of the helix, about 180 Å, allows one to approximate the system as a ferromagnet. The helical order implies that the transition should be very weakly first order even at ambient pressure (Bak and Jensen, 1980). This has indeed been observed (Stishov et al., 2007, 2008; Janoschek et al., 2013). Pfleiderer et al. (1997) found evidence of a strongly first-order transition for pressures $p' < p < p_c$, with $p' \approx 12$ kbar. The tricritical temperature (i.e., the transition temperature at $p = p'$) is $T_{tc} \approx 12$ K. These results were later corroborated by the observation of tricritical wings (Pfleiderer, Julian, and Lonzarich, 2001) (see Fig. 3), and by μSR data that show, for $p' < p < p_c$, phase separation indicative of a first-order transition (Uemura et al., 2006).

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10 We use the residual resistivity, denoted by $\rho_0$, as a measure of quenched disorder. We note that $\rho_0$ is a very rough and incomplete measure of disorder, that many transport theories make simple assumptions regarding the scattering process, and that relating the measured value of $\rho_0$ to theoretical considerations can therefore be difficult. Also, different manifestations of disorder may affect $\rho_0$ differently than they affect magnetism. This may be relevant for certain systems that are nominally rather clean, such as Ni$_3$Pd$_{1-x}$, see Sec. II.C.1.a. Unfortunately, more extensive characterizations of disorder are rarely available.

11 We denote the ferromagnetic transition temperature by $T_C$ irrespective of the order of the transition. In parts of Sec. III, where we want to emphasize that a transition is second order, we denote the critical temperature by $T_C$.

12 The first-order transition at ambient pressure was found by Janoschek et al. (2013) to be of a type that was first predicted by Brazovskii (1975) for different systems. It differs slightly from the phase predicted by Bak and Jensen (1980) for helical magnets.

13 Since the transition is likely to be weakly first order for all $p < p'$, the observed apparent tricritical point separates a very weakly first-order transition from one that is more strongly first order.
TABLE I. Systems showing a first-order transition I: transition-metal and uranium-based compounds. FM = ferromagnetism, SC = superconductivity, $T_C$ = Curie temperature, $T_{ic}$ = tricritical temperature, $\rho_0$ = residual resistivity, and n.a. = not available.

<table>
<thead>
<tr>
<th>System</th>
<th>Order of transition$^a$</th>
<th>$T_C$/$K^b$</th>
<th>Magnetic moment/$\mu_B^c$</th>
<th>Tuning parameter</th>
<th>$T_{ic}$/$K^d$</th>
<th>Wings observed</th>
<th>Disorder$^d$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>MnSi</td>
<td>1st $^{3}$</td>
<td>29.5$^5$</td>
<td>0.4$^6$</td>
<td>Pressure$^e$</td>
<td>$\approx 10^7$</td>
<td>Yes$^4$</td>
<td>0.33$^5$</td>
<td>Weak helimagnet$^{5,6}$</td>
</tr>
<tr>
<td>ZrZn$_2$</td>
<td>1st $^{1}$</td>
<td>28.5$^7$</td>
<td>0.17$^8$</td>
<td>Pressure$^e$</td>
<td>$\approx 5^7$</td>
<td>Yes$^2$</td>
<td>$\geq 0.31^8$</td>
<td>Long history$^7$</td>
</tr>
<tr>
<td>Co$_2$</td>
<td>1st $^{10,11}$</td>
<td>122$^{10,11}$</td>
<td>0.84$^{12}$</td>
<td>Pressure$^{10}$</td>
<td>$\approx 118^10$</td>
<td>(Yes)$^{10}$</td>
<td>0.2–0.6$^{13}$</td>
<td>High $T_{ic}$ and $T_{tc}$</td>
</tr>
<tr>
<td>Ni$_3$Al</td>
<td>(1st?)$^6$</td>
<td>41–15$^6$</td>
<td>0.075$^4$</td>
<td>Pressure$^4$</td>
<td>n.a.</td>
<td>No</td>
<td>0.84$^17$</td>
<td>First-order transition suspected</td>
</tr>
<tr>
<td>UGe$_2$</td>
<td>16,17$^{16}$</td>
<td>52$^{18}$</td>
<td>1.5$^{18}$</td>
<td>Pressure$^{19,18}$</td>
<td>24$^{20}$</td>
<td>Yes$^{18,20}$</td>
<td>0.2$^{19}$</td>
<td>Easy-axis FM coex. FM + SC$^{19}$</td>
</tr>
<tr>
<td>U$_3$P$_4$</td>
<td>11$^1$</td>
<td>138$^{22}$</td>
<td>1.34$^{23}$</td>
<td>Pressure$^{21}$</td>
<td>32$^{21}$</td>
<td>Yes$^{k,21}$</td>
<td>4$^{21}$</td>
<td>Canted easy-axis FM</td>
</tr>
<tr>
<td>URhGe</td>
<td>17,24$^{24}$</td>
<td>9.25$^{25}$</td>
<td>0.42$^{25}$</td>
<td>Pressure$^7$</td>
<td>$\approx 25^7$</td>
<td>Yes$^{24}$</td>
<td>8$^{27}$</td>
<td>Easy-plane FM cox. FM + SC$^{25}$</td>
</tr>
<tr>
<td>UCoGe</td>
<td>17,28$^{28}$</td>
<td>2.5$^{29}$</td>
<td>0.23$^{29}$</td>
<td>None</td>
<td>$&gt;2.5,^m$</td>
<td>No</td>
<td>12$^{29}$</td>
<td>Very weak FM coex. FM + SC$^{29}$</td>
</tr>
<tr>
<td>UCoAl</td>
<td>1a, 30$^{30}$</td>
<td>$0^o^{30,31}$</td>
<td>$0^o^{30,31}$</td>
<td>Pressure$^{26,31}$</td>
<td>$&gt;11,^{30}$</td>
<td>Yes$^{30}$</td>
<td>24$^{30}$</td>
<td>Easy-axis FM</td>
</tr>
<tr>
<td>URhAl</td>
<td>1$^{33}$</td>
<td>$34–25^{32,33}$</td>
<td>$\approx 0.9^{32,33}$</td>
<td>Pressure$^3$</td>
<td>$\approx 11,^{33}$</td>
<td>Yes$^{33}$</td>
<td>$\approx 65^{33}$</td>
<td>Weakly first order</td>
</tr>
</tbody>
</table>

$^a$At the lowest temperature achieved.
$^b$A single value of $T_C$ at the default value of the tuning parameter (ambient pressure, zero field) is given if $T_{ic}$ has also been measured; a range of $T_C$ for a range of control parameters in all other cases.
$^c$For the highest-quality samples.
$^d$Suspected first-order transition near $p = 80$ kbar (Niklowitz et al., 2005; Pfleiderer, 2007).
$^e$Per Ni at $p = 0$ (Niklowitz et al., 2005).
$^f$Per U.
$^g$Via a metamagnetic transition; wings have not been mapped out.
$^h$At the critical pressure $p_c \approx 4$ GPa.
$^i$Pressure decreases $T_C$ (Slooten et al., 2009); TCP is not accessible. $T_C$ increases nonmonotonically upon doping with Rh (Sakarya et al., 2008); order of transition for URh$_x$Co$_{1-x}$Ge not known except for $x = 1$ (second order with $T_C = 9.5$ K).
$^j$Inferred from existence of tricritical wings.
$^k$PM at zero pressure. Uniaxial pressure induces FM, so does doping; see Ishii et al. (2003) and references therein.

$b.$ ZrZn$_2$ crystallizes in the cubic C15 structure and is a true ferromagnet (Matthias and Bozorth, 1958; Pickart et al., 1964) with a small magnetic anisotropy and an ordered moment of 0.17$\mu_B$ per formula unit (Uhlarz, Pfleiderer, and Hayden, 2004). The material can be tuned across the transition by means of hydrostatic pressure. While early experiments (Smith, Mydosh, and Wohlfarth, 1971; Huber, Maple, and Wohlleben, 1975; Grosche et al., 1995) suggested the existence of a quantum-critical point, an increase in sample quality led to the realization that the transition becomes first order near the critical pressure $p_c \approx 16.5$ kbar (Uhlarz, Pfleiderer, and Hayden, 2004). The transition temperature at ambient pressure is $T_C \approx 28.5$ K, and the tricritical temperature is $T_{tc} \approx 5$ K. The phase diagram is qualitatively the same as that shown in Fig. 3; the observation of tricritical wings by Uhlarz, Pfleiderer, and Hayden (2004) confirmed an earlier suggestion by Kimura et al. (2004). The first-order nature of the QPT was confirmed by Kabeya et al. (2012, 2013), who also studied...
TABLE II. Systems showing a first-order transition II: lanthanide-based compounds and strontium ruthenates. $T_C$ = Curie temperature, $T_{ic}$ = tricritical temperature, $\rho_0$ = residual resistivity, and n.a. = not available.

<table>
<thead>
<tr>
<th>System</th>
<th>Order of transition$^a$</th>
<th>$T_C$/K</th>
<th>Magnetic moment/$\mu_B$$^b$</th>
<th>Tuning parameter</th>
<th>$T_{ic}$/K</th>
<th>Wings observed$^c$</th>
<th>Disorder$^d$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>La$_{1-x}$Ce$_x$In$_2$</td>
<td>1st$^4$</td>
<td>22–19.5$^{1,3}$</td>
<td>n.a.</td>
<td>Composition$^1$</td>
<td>&gt;22$^f$</td>
<td>No</td>
<td>n.a.</td>
<td>Third phase?</td>
</tr>
<tr>
<td>SmNiC$_2$</td>
<td>1st$^2$</td>
<td>17–15$^{1,2}$</td>
<td>0.32$^2$</td>
<td>Pressure$^2$$^4,5$</td>
<td>&gt;17?</td>
<td>No</td>
<td>2</td>
<td>Other phases$^2$</td>
</tr>
<tr>
<td>YbCu$_2$Si$_2$</td>
<td>1st$^3$</td>
<td>4.7 – 3.5$^{3,2}$</td>
<td>0.16–0.42$^{3,3}$</td>
<td>Pressure$^3$$^4,5$</td>
<td>n.a.</td>
<td>No</td>
<td>&lt;0.5$^b$</td>
<td>Strong Ising anisotropy$^3$</td>
</tr>
<tr>
<td>YbIr$_2$Si$_2$</td>
<td>1st$^7$</td>
<td>2.3 – 1.3$^4$</td>
<td>n.a.</td>
<td>Pressure$^7$</td>
<td>n.a.</td>
<td>No</td>
<td>22$^j$</td>
<td>First-order transition suspected</td>
</tr>
<tr>
<td>CePt</td>
<td>1st$^8$</td>
<td>5.8 – 0$^8$</td>
<td>n.a.</td>
<td>Pressure$^8$</td>
<td>n.a.</td>
<td>No</td>
<td>1$^l$</td>
<td>FM order suspected$^l$</td>
</tr>
<tr>
<td>Sr$_{1-x}$Ca$_x$RuO$_3$</td>
<td>1st$^{10}$</td>
<td>160 – 0$^k$</td>
<td>0.8 – 0$^k$</td>
<td>Composition$^{10}$</td>
<td>n.a.</td>
<td>Yes$^{11}$</td>
<td>n.a.</td>
<td>Ceramic samples</td>
</tr>
<tr>
<td>Sr$_3$Ru$_2$O$_7$</td>
<td>1st$^l$</td>
<td>0$^m$</td>
<td>0$^m$</td>
<td>Pressure$^m$</td>
<td>n.a.</td>
<td>No</td>
<td>&lt;0.5$^{11}$</td>
<td>Foliated wings</td>
</tr>
</tbody>
</table>

aAt the lowest temperature achieved.
bPer formula unit unless otherwise noted.
cFor the highest-quality samples.
dFor $x = 1.0–0.9$.
eFirst order for $x = 1$, TCP not accessible.
fFor $p = 0–2$ GPa.
gFor pressures $p \approx 11.5 – 9.4$ GPa.
hFor pressures $p = 9.4–11.5$ GPa.
iFor pressures $p \approx 10 – 8$ GPa.
 jFor a magnetic sample at pressures $p \approx 8–10$ GPa. Samples with $\rho_0$ as low as 0.3 $\mu$Ω cm at ambient pressure have been prepared (Yuan et al., 2006).
kFor $x = 0$ to $x \approx 0.7$.
lPhase diagram not mapped out completely; the most detailed measurements show tips of wings. See Wu et al. (2011).
mParamagnetic at ambient pressure. Hydrostatic pressure drives the system away from FM, and uniaxial stress drives it toward FM. See Wu et al. (2011) and references therein, especially Ikeda et al. (2000).

8Larrea et al. (2005).

c. CoS$_2$

CoS$_2$ crystallizes in a cubic pyrite structure. It is an itinerant ferromagnet with $T_C$ $\approx$ 124 K, an ordered moment of 0.84$\mu_B$/Co, and an effective moment of 1.76$\mu_B$/Co (Jarrett et al., 1968; Adachi, Sato, and Takeda, 1969). Density-functional calculations concluded that CoS$_2$ is a half-metallic ferromagnet (Zhao, Callaway, and Hayashibara, 1993; Mazin, 2000). The spin polarization is high at about 56% (Wang, Chen, and Leighton, 2004), and the transport coefficients and the thermal expansion coefficient show unusual behavior in the vicinity of the transition (Yomo, 1979; Adachi and Ohkohchi, 1980). Magnetization measurements indicate that the transition is almost first order at ambient pressure (Wang, Chen, and Leighton, 2004). Hydrostatic pressure decreases $T_C$, and at a pressure of about 0.4 GPa the nature of the transition changes from second order to first order, with a tricritical temperature $T_{ic}$ $\approx$ 118 K (Goto et al., 1997). A much lower value for the tricritical pressure was found by Otero-Leal et al. (2008); however, this analysis depended on a specific model equation of state. Sidorov, Krasnorussky et al. (2011) confirmed a strongly first-order QPT at a critical pressure of about 4.8 GPa. $T_C$ is also suppressed if selenium is substituted for sulfur, and the transition again becomes first order at a small selenium concentration, with 1% of selenium roughly equivalent to a pressure of 1 GPa (Hiraka and Endoh, 1996).

Two groups have investigated the $p$-$T$ phase diagram at higher pressures up to the QPT: Barakat et al. (2005) observed a monotonically decreasing $T_C$ with increasing pressure. They inferred a first-order quantum phase transition at $p_c$ $\approx$ 6 GPa from a change of the temperature dependence of the resistivity [$\rho(T) = \rho_0 + A T^n$] from $n = 2$ in the FM phase to $n \approx 1.6$ for $p > p_c$. Their samples had a residual resistivity $\rho_0$ $\approx$ 2 $\mu$Ω cm and a residual resistance ratio (RRR) of about 60. Sidorov, Krasnorussky et al. (2011) performed experiments on a cleaner sample ($\rho_0$ $\approx$ 0.7 $\mu$Ω cm) and concluded that $p_c$ $\approx$ 4.8 GPa. They found that the temperature dependence of the resistivity does not change across the transition, with $n = 2$ both below and above $p_c$, while the residual resistivity drops by about a factor of 3 as the transition is crossed.

These discrepancies notwithstanding, all experiments agree on the first-order nature of the quantum phase transition. This makes the phase diagrams of CoS$_2$, ZrZn$_2$, and MnSi qualitatively the same.
d. Ni$_3$Al

Ni$_3$Al crystallizes in the simple cubic Cu$_3$Au structure. Its magnetic properties depend on the exact composition; the stoichiometric compound at ambient pressure is a ferromagnet with $T_C = 41$ K and a small ordered moment of 0.075μ$_B$/Ni (de Boer et al., 1969; Niklowitz et al., 2005). $T_C$ decreases upon the application of hydrostatic pressure and vanishes at a critical pressure of 8.1 GPa (Niklowitz et al., 2005). The resistivity of stoichiometric Ni$_3$Al shows a pronounced NFL temperature dependence on either side of the transition, $\Delta \rho \propto T^n$, with $n$ somewhere between 3/2 and 5/3 (Fluitman et al., 1973; Steiner et al., 2003; Pfleiderer, 2007). At ambient pressure and in zero magnetic field Steiner et al. (2003) found $n = 1.65$ for temperatures between about 0.5 and 3.5 K. The prefactor is comparable with that of the $T^{3/2}$ behavior of the resistivity in ZrZn$_2$ (Pfleiderer et al., 2001; Yelland et al., 2005).

The transition at ambient pressure is second order, and the overall form of the phase diagram is consistent with the results of the spin-fluctuation theory described in Sec. III.C.2, as is the logarithmic temperature dependence of the specific heat (Sato, 1975; Niklowitz et al., 2005; Yang et al., 2011). However, studies of the temperature dependence of the resistivity under pressure suggest that the QPT at the critical pressure is first order (Niklowitz et al., 2005; Pfleiderer, 2007). This is analogous to the behavior of MnSi; see Sec. II.B.1.a.

$T_C$ also decreases upon doping Ni$_3$Al with Pd (Sato, 1975) or Ga (Yang et al., 2011); these systems are discussed in Sec. II.C.1.b.

2. Uranium-based compounds

Ferromagnetism with a first-order transition at low temperatures has been observed in the uranium-based heavy-fermion (HF) compounds UGe$_2$ (Huxley, Sheikin, and Braithwaite, 2000; Taufour et al., 2010; Kotegawa, Taufour et al., 2011), URhGe (Huxley et al., 2007), and UCoGe (Hattori et al., 2010). UCoAl is paramagnetic at ambient pressure, but very close to a first-order QPT (Aoki, Combier et al., 2011). The ferromagnetism is due to 5f electrons. The extent to which these electrons are localized or itinerant, and the consequences for neutron scattering observations, have been investigated in some detail (Yaouanc et al., 2002; Fujimori et al., 2012; Chubukov, Betouras, and Efremov, 2014). Coexistence of ferromagnetism and superconductivity was found in UGe$_2$ (Saxena et al., 2000; Huxley et al., 2001), URhGe (Aoki et al., 2001), and UCoGe (Huy, Gasparini, de Nijs et al., 2007); for a recent overview, see Aoki and Flouquet (2014).

a. UGe$_2$

UGe$_2$ crystallizes in an inversion-symmetric orthorhombic structure, and the best samples have residual resistivities as low as 0.2 μΩ cm (Saxena et al., 2000). Taufour et al. (2010) found the residual resistivity to be strongly pressure dependent. The Curie temperature at ambient pressure is $T_C \approx 52$ K (Saxena et al., 2000; Aoki et al., 2001; Huxley et al., 2001; Aoki and Flouquet, 2012). $T_C$ decreases with increasing hydrostatic pressure and vanishes at $p \approx 16$ kbar, which coincides with the pressure where the superconductivity disappears. Within the ferromagnetic phase a further transition is observed, across which the magnitude of the magnetic moment changes discontinuously. The associated transition line starts near the peak in the superconducting transition temperature, ends at a critical point at a temperature of about 4 K, and is replaced by a crossover at higher temperatures (Huxley et al., 2007; Taufour et al., 2010); see Fig. 5. The tricritical temperature has been measured to be $T_{tc} \approx 24$ K.

FIG. 3. Phase diagram of MnSi. In the temperature-pressure ($T$-$p$) plane the transition temperature drops from $T_C = 29.5$ K at ambient pressure and changes to second to first order at $p^* = 12$ kbar, where $T_C \approx 12$ K, $T_C$ vanishes at $p_e = 14.6$ kbar. In the magnetic field, pressure ($B$-$p$) plane at $T = 0$, and everywhere across the shaded wing, the transition is first order up to a “critical end point” (see footnote) estimated to be located at $B_m = 0.6$ T and $p_m = 17$ kbar. From Pfleiderer, Julian, and Lonzarich, 2001.

FIG. 4. μSR results for the volume fraction with static magnetic order. The nonzero volume fraction less than unity at $T = 0$ for intermediate pressures indicates phase separation, which in turn is indicative of a first-order transition. From Uemura et al., 2007.
Hydrostatic pressure decreases the magnetic moments, which are partially delocalized in these materials, the ordered moment is often reduced compared to the free ion value and an enhanced electronic specific heat is observed. In addition, they are characterized by a strong Ising anisotropy (Sechovsky and Havela, 1998). Two main mechanisms control the delocalization of the $5f$ electrons and thus the magnetism: the direct overlap of neighboring U $5f$ orbitals, and their hybridization with the $d$ electrons. For inter-U distances smaller than the Hill limit ($d_{U-U} \approx 3.4-3.6$ Å (Hill, 1970), the strong overlap of the $5f$ orbitals results in a nonmagnetic ground state. Larger values yield a FM or AFM ordered ground state. For values close to this limit the $f-d$ hybridization strength controls the magnetic properties. There is a clear tendency of these systems to show magnetic order with increasing $d$-electron filling of the $T$ element (Sechovsky and Havela, 1998). The strongest electronic correlations are therefore found in UTX compounds with intermediate values of $d_{U-U}$ and $d$-electron filling.

$\text{URhGe}$ has a $d_{U-U} = 3.5$ Å close to the Hill limit. It is ferromagnetic with a Curie temperature $T_C = 9.5$ K and an ordered moment of $0.42\mu_B$, oriented along the $c$ axis. A magnetic field parallel to the $b$ axis suppresses $T_C$ and leads to a tricritical point at $T \approx 1$ K and $H_b \approx 12$ T (Huxley et al., 2007). With an additional field in the $c$ direction, tricritical wings appear; see Fig. 6. The superconductivity that is observed in zero field (Aoki et al., 2001) is absent at intermediate fields, but reappears at low temperatures in the vicinity of the tricritical wings (Levy et al., 2005; Huxley et al., 2007).

The nature of the magnetic order in $\text{UCoGe}$, ferromagnetic or otherwise, was initially unclear. This, together with the observation that $\text{URhGe}$ is ferromagnetic, prompted the study of $\text{URh}_{1-x}\text{Co}_x\text{Ge}$ alloys (Sakarya et al., 2008), and the final conclusion was that $\text{UCoGe}$ is indeed a weak ferromagnet with a Curie temperature near 3 K and a small ordered moment of 0.03$\mu_B$ (Huy, Gasparini, de Nijs et al., 2007). The transition was found to be weakly first order by means of nuclear quadrupole resonance measurements (Hattori et al., 2010). Hydrostatic pressure decreases $T_C$ (Hassinger et al., 2008; Slooten et al., 2009) which vanishes near the maximum of the superconducting dome; see Fig. 7. A tricritical point must appear as $T_C$ increases upon doping with Rh, see Fig. 8, but the order of the transition has not been studied as a function of pressure and magnetic field.

Both of these materials belong to the ternary UTX intermetallic U compounds where T is one of the late transition metals and X is a $p$-electron element. They crystallize in the orthorhombic TiNiSi structure (space group $P_{nma}$). For lattice parameters, see Troč and Tran (1988) and Canepa et al. (1996). Because the $5f$ electrons, which carry the magnetic moments, are partially delocalized in these regions, the transition between the phases FM1 and FM2, and the tricritical point (TCP). The critical point marked CEP (see footnote 6) is related to the transition between the phases FM1 and FM2. From Taufour et al., 2010.

b. $\text{U}_3\text{P}_4$

$\text{U}_3\text{P}_4$ at ambient pressure is a ferromagnet with $T_C = 138$ K (Trzebiatowski and Troč, 1963). It crystallizes in a bcc structure with no inversion symmetry, and the magnetic structure is canted with a FM component along (111) (Heimbrecht, Zumbusch, and Blitz, 1941; Zumbusch, 1941; Burlet et al., 1981; Wiśniewski, Gukasov, and Henkie, 1999). Pressure reduces $T_C$ until a QPT is reached at $p_c \approx 4$ GPa. From measurements of the resistivity and the magnetic susceptibility at $p \approx 1.5$ GPa, Araki et al. (2015) concluded that the transition changes from second order to first order with a tricritical temperature $T_{tc} = 32$ K. Consistent with this, the pressure dependence of $T_C$ changes from a Hertz-type $T_C \propto (p - p_c)^{3/4}$ behavior to $T_C \propto (p - p_c)^{1/2}$. In a magnetic field, metamagnetic behavior has been observed that is indicative of tricritical wings, although the wings have not been mapped out.

c. $\text{URhGe}$ and $\text{UCoGe}$

Both of these materials belong to the ternary UTX intermetallic U compounds where T is one of the late transition metals and X is a $p$-electron element. They crystallize in the orthorhombic TiNiSi structure (space group $P_{nma}$). For lattice parameters, see Troč and Tran (1988) and Canepa et al. (1996). Because the $5f$ electrons, which carry the magnetic moments, are partially delocalized in these regions, the transition between the phases FM1 and FM2, and the tricritical point (TCP). The critical point marked CEP (see footnote 6) is related to the transition between the phases FM1 and FM2. From Taufour et al., 2010.

FIG. 5. Phase diagram of UGe$_2$ in the temperature-pressure plane. Shown are the paramagnetic (PM) phase, two ferromagnetic phases (FM1 and FM2), and the tricritical point (TCP). The critical point marked CEP (see footnote 6) is related to the transition between the phases FM1 and FM2. From Taufour et al., 2010.

FIG. 6. Phase diagram of URhGe in the space spanned by temperature and magnetic fields in the $b$ and $c$ directions. The dark shaded regions indicate the presence of superconductivity. From Huxley et al., 2007.
function of the Rh concentration. Similarly, in pure UCoGe tricritical wings should appear in a magnetic field, analogously to what is observed in UCoAl, see Fig. 10. A recent study reported that $T_C$ is suppressed by doping with Ru, with an extrapolated critical Ru concentration of about 31% (Vališka et al., 2015). The order of the transition has not been determined.

UCoGe displays the coexistence of superconductivity and ferromagnetism below 0.8 K (Huy, Gasparini, de Nijs et al., 2007; Slooten et al., 2009). In contrast to both UGe$_2$ and URhGe the superconductivity is observed in both the ferromagnetic and paramagnetic phases; see Fig. 7.

d. UCoAl

At ambient pressure and zero field, UCoAl is a paramagnet with a strong uniaxial magnetic anisotropy (Sychrovsky et al., 1986). It crystallizes in the hexagonal ZrNiAl structure consisting of U-Co and Co-Al layers that alternate along the $c$ axis. The inter-U distance is $d_{U-U} \approx 3.5$ Å (the same value as in URhGe, see II.B.2.c), but a large $d$ filling leads to UCoAl being paramagnetic (Sychrovsky and Havela, 1998). Its isoelectronic analog URhAl is ferromagnetic with $d_{U-U} \approx 3.63$ Å (cf. Sec. II.B.2.e). These observations suggest that UCoAl is close to a FM instability, which is indeed the case: Application of a magnetic field along the easy magnetization axis (the crystallographic $c$ axis) induces a first-order metamagnetic phase transition at $H_m \approx 0.7$ T at low temperature with an induced moment of about $0.3\mu_B$ (Andreev et al., 1985; Mushnikov et al., 1999). Moreover, uniaxial stress induces ferromagnetism (Ishii et al., 2003; Shimizu, Salce et al., 2015). The susceptibility shows Curie-Weiss behavior for $T > 40$ K with a fluctuating moment of about $1.6\mu_B$, much larger than the induced moment of $0.3\mu_B$ (Havela et al., 1997). The magnetism is believed to be itinerant with the U 5f electrons providing the main contribution (Eriksson, Johansson, and Brooks, 1989; Wulf et al., 1990; Mushnikov et al., 1999); polarized-neutron diffraction experiments have found the magnetic moment exclusively at the U sites with the orbital moment being twice as large as (and antiparallel to) the spin moment (Wulf et al., 1990; Javorský et al., 2001).

Studies of the magnetostriction, magnetoresistivity (Aoki, Combier et al., 2011), nuclear magnetic resonance (Karube et al., 2012), and thermopower (Palacio-Morales et al., 2013) indicate that the field-induced first-order transition terminates in a critical point at a temperature $T_0 = 11$ K at ambient pressure, as illustrated in Fig. 9: $\Delta L(H)/L$ shows a steplike jump at $H_m$ for $T < T_0$ which becomes smooth for $T \geq T_0$. A determination of critical exponents suggests that the transition at $T_0$ is in the 3-d Ising universality class (Karube et al., 2012). $H_m$ increases with pressure and each wing terminates in a quantum-critical point (denoted by QCEP in the figure) at $P \approx 1.5$ GPa and $\mu_BH \approx 7$ T. At the wing-tip point a pronounced enhancement of the effective mass (derived from the coefficient of the $T^2$ term in the electrical resistivity) is observed (Aoki, Combier et al., 2011).

The resulting $T$-$P$-$H$ semischematic phase diagram is shown in Fig. 10, which demonstrates the presence of
tricritical wings in UCoAl. The red dots represent the experimental values for $H_m$ determined by magnetoresistivity (with $J \perp H$) and magnetostriction measurements. Since $T_0 = 11$ K at the ambient pressure, the tricritical point (TCP) must be located at $T > 11$ K. At pressures higher than 1.5 GPa, the first-order character of the metamagnetic transition disappears and new features in the form of kinks in the magnetoresistivity and Hall effect are observed at $H_m$ and $H^*$ (Combier et al., 2013). Investigations of the transverse and longitudinal resistivities and of the magnetization under pressure (Combier, 2013) point to a much richer phase diagram, where the exact location of the QCEP remains uncertain, with possible changes of the Fermi surface as well as the appearance of new phases around the QCEP.

The substitution of Fe for Co was found to lead to a FM ground state in zero field and ambient pressure by Karube et al. (2015). By nuclear quadrupole resonance measurements they found a first-order transition in $U(\text{Co}_{1-x}\text{Fe}_x)\text{Al}$ with a $T_C$ of about 10 and 17 K for $x = 0.1$ and 0.02, respectively.

e. URhAl

URhAl belongs to the same UTX compound family as URhGe, UCoGe, and UCoAl. It has the same layered hexagonal ZnNiAl-type crystal structure as UCoAl, but with $d_{\text{U-U}} = 3.63$ Å, larger than the Hill limit (cf. Sec. II.B.2.c). Consistent with this, and contrary to UCoAl which has a nonmagnetic ground state, URhAl orders ferromagnetically via a second-order transition. Values of $T_C$ between 27 and 34 K have been reported, with strong Ising-like ordered moments of 0.9$\mu_B$/U along the $c$ axis (Veenhuizen et al., 1988; Combier, 2013; Shimizu et al., 2015b).

The itinerant versus localized nature of magnetism in URhAl is controversial, as it is in many other UTX compounds. A peak at 380 meV in inelastic neutron scattering experiments (Hiess et al., 1997) was interpreted as indication of an intermultiplet transition, suggesting 5$f$-electron localization. X-ray magnetic circular dichroism experiments also indicate a high degree of localization of the 5$f$ electrons (Grange et al., 1998). On the other hand, polarized-neutron studies point to a rather strong delocalization of the 5$f$ electrons (Paixao et al., 1992). Moreover, band-structure calculations based on an itinerant approach can reproduce most of the experimental findings (Kučera et al., 1998; Kuneš et al., 2001).

Pressure experiments were performed on a rather clean single crystal with a $T_C \approx 14$ and $T_C = 28$ K (Combier, 2013). At ambient pressure the phase transition is mean field like characterized by a single peak in $C/T$ and a kink in the thermal expansion ratio $\Delta L/L$. The magnetization with $H//c$ shows a clear hysteresis at 2 K with a remanent magnetization of 0.9$\mu_B$/U. Transport experiments on moderately disordered samples ($\rho_0 \approx 65 \mu \Omega \text{cm}$ near the transition) have mapped out the phase diagram in more detail (Shimizu et al., 2015a, 2015b). The QPT at a critical pressure $p_c \approx 5.2$ GPa is weakly first order, allowing strong spin fluctuations to be observed in

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{fig10}
\caption{\emph{T}-\emph{P}-\emph{H} phase diagram of UCoAl. The tricritical wings are determined by the observation of a first-order metamagnetic transition at $H_m$ (red dots); they are bounded by lines of second-order transitions at $T_0$ and end in quantum "critical end points" (QCEPs) (see the explanation in the Fig. 1 caption). The critical pressure $P_c$ is negative and the tricritical point (TCP) is not accessible. From Aoki, Combier et al., 2011.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.9\textwidth]{fig11}
\caption{Upper panel: Temperature-pressure phase diagram of URhAl in zero field determined from resistivity measurements. Lower panel: Temperature-pressure-field diagram inferred from metamagnetic behavior observed in an external field. From Shimizu et al., 2015b.}
\end{figure}
transport and thermodynamic properties. Metamagnetic signatures in a magnetic field imply the existence of tricritical wings; see Fig. 11.

3. Lanthanide-based compounds

a. La1-xCe1xIn2

CeIn2 crystallizes in the orthorhombic CeCu2 structure and undergoes a first-order transition to a FM state at \( T_C = 22 \text{ K} \) (Rojas et al., 2009). This conclusion on the basis of discontinuities at \( T_C \) in the resistivity, the thermal expansion, and the magnetic entropy was later corroborated by \( \mu \)SR measurements (Rojas et al., 2011). Application of hydrostatic pressure increases \( T_C \) (Mukherjee, Iyer, and Sampathkumaran, 2012), but upon doping with lanthanum \( T_C \) decreases, to about 19.4 K in \( \text{La}_1-x\text{Ce}_x\text{In}_2 \) with \( x = 0.9 \), and the transition remains first order (Rojas et al., 2011). The same \( \mu \)SR measurements indicated the existence of a second magnetic phase with long-range order in between the FM and PM phases. The nature of this phase is not known. Doping with Ni decreases \( T_C \) sharply, and the transition in \( \text{Ce}((\text{In})_{1-x}\text{Ni}_x)_2 \) has been reported to be second order to a FM for \( x = 0.025, 0.05 \), and 0.15 (Rojas, Espeso, and Fernández, 2013). However, an earlier experiment by Sung et al. (2009) concluded that the ground state for \( x = 0.15 \) is AFM.

b. SmNiC2

The ferromagnetic charge-density-wave (CDW) compound SmNiC2 has a \( T_C \) of about 17 K which is weakly susceptible to pressure (Woo et al., 2013). The polycrystalline samples measured had a residual resistivity of less than 2 \( \mu \Omega \text{cm} \) for pressures below about 3 GPa. The PM-FM transition is first order and remains first order as the pressure is increased from 0 to 2 GPa, with \( T_C \) dropping to 15 K. At higher pressure, there is a second-order or weakly first-order transition from the FM to a phase of unclear nature, and at least two other phases appear at low temperature. Since the nonmagnetic phase in this material is a CDW state below \( T \approx 150 \text{ K} \), the phase diagram may fall outside the classification provided by Fig. 2 and the first-order transition may be of different origin than in other materials; see Sec. III.F.

c. Yb-based systems

YbCu2Si2 crystallizes in the body-centered ThCr2Si2 structure and does not order magnetically at ambient pressure. A transition to a magnetically ordered phase under pressure was suggested on the basis of transport measurements (Alami-Yadri and Jaccard, 1996; Alami-Yadri, Wilhelm, and Jaccard, 1998), and later confirmed by means of Mössbauer data (Winkelmann et al., 1999). Fernandez-Pañella et al. (2011) concluded from susceptibility measurements that the nature of the ordered phase is FM, and the transition is likely first order (Winkelmann et al., 1999; Colombier et al., 2009; Fernandez-Pañella et al., 2011). The FM nature of the ordered phase was confirmed by Tateiwa et al. (2014), who also found evidence for phase separation indicative of a first-order transition.

YbIr2Si2 crystallizes in either the ThCr2Si2 structure or the \( P \)-type \( \text{CaBe}_2\text{Ge}_2 \) structure, depending on the synthesis conditions (Hossain et al., 2005). The former is magnetically (presumably AFM) ordered below 0.7 K, whereas the latter is a paramagnet at ambient pressure. Yuan et al. (2006) found that by applying pressure the system in its \( P \)-type structure undergoes a first-order QPT to an ordered phase at a critical pressure \( p_c \approx 8 \text{ GPa} \). The nature of the ordered phase is suspected to be FM, but additional investigations are needed. Recent measurements of the resistivity under hydrostatic pressure as high as 15 GPa found NFL behavior in a pressure range \( 3 \leq p \leq 8 \text{ GPa} \), and confirmed the sudden appearance of magnetic order at 8.3 GPa, suggesting a first-order QPT (Macovei, 2010). The transition temperature shifts to higher values and shows a weak maximum around 11 GPa, a behavior very similar to that of YbRh2Si2 under pressure (Mederle et al., 2001; Knebel et al., 2006). YbRh2Si2 evolves from an AFM to a FM ordered state under chemical pressure (Co substitution) (Lausberg et al., 2013) and possibly even under hydrostatic pressure (Knebel et al., 2006). This suggests that the nature of the magnetic ordered phase in YbIr2Si2 could also be AFM, but more investigations are needed.

d. CePt

CePt under pressure has been reported to display a FM QPT at \( p_c \approx 12.1 \text{ GPa} \) (Larrea et al., 2005). The transition at \( p = 0 \) is second order (Holt et al., 1981). No magnetization measurements have been performed under pressure. The FM signature is strongly weakened under pressure well before \( p_c \) and transport experiments indicate a sudden drop of the phase boundary line close to \( p_c \), suggesting the presence of a first-order transition.

4. Strontium ruthenates

The perovskite ruthenates, which include \( \text{Sr}_2\text{RuO}_4 \) and \( \text{Sr}_4\text{Ru}_2\text{O}_{10} \) in addition to \( \text{SrRuO}_3 \) and \( \text{Sr}_3\text{Ru}_2\text{O}_7 \), belong to the Ruddlesden-Popper series; for a historical overview, see Mackenzie and Grigera (2004). In \( \text{SrRuO}_3 \), a QPT can be triggered by means of doping with calcium, whereas the phase diagram of \( \text{Sr}_2\text{Ru}_2\text{O}_7 \) has been explored by applying pressure and an external magnetic field. In \( \text{Sr}_{1-x}\text{Ca}_x\text{RuO}_3 \) a variety of very different behaviors has been observed, which is likely due to different sample preparation methods (bulk ceramic, bulk powder, and thin films). We therefore discuss this material both in the present section and in Secs. II.C.1 and II.E.3.

a. \( \text{Sr}_{1-x}\text{Ca}_x\text{RuO}_3 \) (bulk ceramic samples)

\( \text{Sr}_{1-x}\text{Ca}_x\text{RuO}_3 \) is a metallic system that crystallizes in an orthorhombically distorted perovskite structure. \( \text{SrRuO}_3 \) is an itinerant ferromagnet with a second-order transition at \( T_C \approx 160 \text{ K} \) (Kim et al., 2003) and an ordered moment of about \( 1 \mu_\text{B}/\text{Rh} \), while \( \text{CaRuO}_3 \) is a strongly exchange-enhanced Pauli paramagnet with no sign of metamagnetism and a Fermi-liquid ground state with an anomalously low coherence scale (Schneider et al., 2014). Long-range FM order disappears for a Ca concentration around \( x \approx 0.7 \), and nuclear magnetic resonance (NMR) experiments established the presence of FM spin fluctuations for all concentrations, the Curie-Weiss behavior of the susceptibility with a Weiss
temperature that changes sign at \( x_c \) notwithstanding (Yoshimura et al., 1999). This and the large effective moment (compared to the ordered one) of about \( 3\mu_B/Ru \) seemed to make Sr\(_{1-x}\)Ca\(_x\)RuO\(_3\) a good candidate for the SCR theory of itinerant ferromagnetism (cf. Sec. I.B). However, a \( \mu \)SR study by Uemura et al. (2007) of ceramic samples with \( x = 0.65 \) and 0.7 found a finite volume fraction of ferromagnetic order and a suppression of the critical dynamics at smaller values of \( x \). These results are similar to the corresponding ones in MnSi (Fig. 4) and are indicative of a first-order transition. No information about the disorder strength in these samples is available. For bulk powder samples and epitaxial thin films of the same material rather different results have been obtained; see Secs. II.C.1.g and II.E.3, respectively.

b. Sr\(_2\)Ru\(_3\)O\(_7\)

Very clean samples of Sr\(_2\)Ru\(_3\)O\(_7\) have been prepared, with residual resistivities of less than 0.25 \( \mu \Omega \)cm (Perry and Maeno, 2004). The ground state in zero field and at ambient pressure is PM close to a FM instability (Ikeda et al., 2000). In the generic phase diagram of Fig. 2(a) this places the system between the tricritical wings (see Fig. 12), as is the case for UCoAl, Fig. 10. A magnetic field applied in the magnetically easy \( ab \) plane takes the system through the metamagnetic wings at about 5 T if the temperature is low enough; see Fig. 12, inset (i). Hydrostatic pressure and uniaxial stress drive the system away from and toward ferromagnetism, respectively (Ikeda et al., 2001, 2004; Chiao et al., 2002; Wu et al., 2011). Wu et al. (2011) investigated the ac susceptibility under pressure across the metamagnetic transition. They found a QCP (denoted by QCEP in Fig. 12) (see the explanation in the Fig. 1 caption) at \( p_\text{c} \approx 13.6 \) kbar, but no divergence of the susceptibility at this point as would be expected for the generic model of quantum-critical metamagnetism (Millis et al., 2002), implying that the metamagnetism cannot solely be explained by field-induced ferromagnetism.

Another way to navigate the phase diagram is to change the field direction out of the magnetically easy \( ab \) plane: Changing the field tilt angle \( \theta \) allows one to follow the wings and suppress the critical temperature \( T^* \) that marks the top of the wing (Grigera et al., 2001, 2003). As \( T^* \) goes to zero, a second sheet of the wing appears, and instead of the QCP that is observed in simpler systems (see Sec. II.B.2) a more complicated phase structure emerges (Grigera et al., 2004; Perry et al., 2004; Rost et al., 2011). The observed bifurcation of the wings has been modeled phenomenologically by means of a Landau theory (Green et al., 2005). The phase between the two sheets, which is observed with field tuning but not with pressure tuning (Wu et al., 2011), has been interpreted as a magnetic nematic (i.e., a non-\( s \)-wave FM) (Grigera et al., 2004; Borzi et al., 2007; Raghu et al., 2009; Rost et al., 2011; Stingl et al., 2011), or an inhomogeneous phase analogous to the Fulde-Ferrell-Larkin-Ovchinnikov phase in superconductors (Berridge et al., 2009, 2010), but the details are not well understood. Magnetic neutron scattering experiments have identified an incommensurate SDW order with an ordered \( q \) in the phase between the sheets, and an additional phase at slightly higher magnetic fields with a different ordering wave vector \( q = (0.218, 0, 0) \) (Lester et al., 2015).

5. Discussion, and comparison with theory

A striking aspect of the phase diagrams discussed in this section is their universality. As illustrated in Tables I and II and discussed in that context, phase diagrams that are qualitatively the same as the one shown in Fig. 1 are observed in a wide variety of systems with very different electronic
structures and different symmetries of the order parameter. Their only commonality is that they are metallic ferromagnets with rather small amounts of disorder (see footnote 10). This universal behavior calls for an equally universal explanation of the first-order nature of the QPT. Although quantitative modeling of the phase diagram is still lacking, the theory described in Sec. III.B.2 can explain the phase diagram qualitatively in terms of a fluctuation-induced first-order transition, with generic soft modes in the conduction-electron system playing the role of the extraneous (to the order parameter) soft modes that drive the transition first order.

There are large quantitative differences between the systems listed in Tables I and II. Sang, Belitz, and Kirkpatrick (2014) showed that the sizes of the tricritical points, which vary dramatically from material to material, correlate with the saturation magnetization as expected from the theory discussed in Sec. III.B.2. Regarding the shape of the wings, theory and all experiments agree that the wings point in the “forward” direction, i.e., the wing tips are located at a larger value of the control parameter than the first-order transition in zero field. However, the curvature of the wings is not expected to be universal; it depends on the relation between the experimental control parameter and the theoretical one (i.e., the mass term in a LGW theory), which in turn depends on microscopic details. For instance, the wings in UGe$_2$, Fig. 1, have a pronounced curvature, whereas the ones in UCoAl, Fig. 10, are almost flat. Similarly, the shape of the lines that connect the tricritical point with the wing tips is not universal. Wysokiński, Abram, and Spalek (2014, 2015) considered a model containing $f$ electrons in addition to conduction electrons and achieved good agreement with the shape of the wings in UGe$_2$. The physical mechanism that leads to a first-order QPT in their theory is the same as the one discussed in Sec. III.B.2.

There also is a clear correlation between the size of the ordered moment and the value of $T_{\text{tc}}$; see Tables I and II. This is consistent with the theory, which predicts that $T_{\text{tc}}$ is proportional to the ordered magnetic moment (for given microscopic temperature and magnetic-moment scales, which one would expect to be similar for systems that are chemically similar) (Belitz, Kirkpatrick, and Rollbühler, 2005). For instance, within the uranium-based systems there is a rough correlation between $T_{\text{tc}}$ and the ordered moment. A U-based system in which no first-order transition has been found is UIr; see Sec. II.C.1 and Table III. Since the ordered moment in the phase FM3 of UIr, from which the QPT to the PM phase occurs, is smaller than the one in UGe$_2$ by more than a factor of 30, and smaller than the one in URhGe by a factor of more than 10 (Kobayashi et al., 2006), one expects $T_{\text{tc}}$ to be smaller by a similar factor. This would put $T_{\text{tc}}$ well below 1 K, and possibly lower than 100 mK, which is less than the lowest $T_{\text{tc}}$ observed in UIr. Similarly, in the first group of materials in Tables I and II $T_{\text{tc}}$ correlates with the size of the ordered moment: CoS$_2$ has the largest moment and the highest $T_{\text{tc}}$, while in Ni$_3$Al, which has the smallest moment, a first-order transition at very low temperatures is suspected but has not yet been convincingly observed. More generally, it is conceivable that $T_{\text{tc}}$ in several weakly disordered systems is rather low and has not been observed so far. A related issue is the robustness of the first-order transition discussed in Secs. III.B.2 and IV.A.

C. Systems showing a continuous transition

We now discuss systems that show a continuous transition at low temperatures; their properties are summarized in Tables III, IV, and V. Most of these materials are composition tuned, which introduces various amounts of disorder, and they can be classified with respect to the disorder strength. The first group is known or suspected to be relatively weakly disordered as judged by the residual resistivity (see footnote 11); see Table III. Consistent with this, their phase diagrams have the shape shown in Fig. 2(b). In the second group, Table IV, the disorder is strong, and the phase diagrams are of the form shown in Fig. 2(d). YbNi$_2$P$_2$ falls into a separate category due to its quasi-one-dimensional electronic structure which sets it apart from all other materials we discuss; see Table V.

1. Weakly disordered systems

a. Ni$_3$Pd$_{1-x}$

NiPd alloys, which crystallize in an fcc structure, form a series of solid solutions whose composition can be varied continuously from pure Pd to pure Ni. The alloying procedure can produce very little disorder as measured by the residual resistivity $\rho_0$, which has been reported not to exceed 0.1 $\mu\Omega$cm for any concentration (Ikeda, 1987). A small concentration (about 2.5%) of Ni induces FM order (Murani, Tari, and Coles, 1974). This composition-induced QPT was studied by Nicklas et al. (1999) by means of heat capacity, electrical resistivity, and magnetization measurements. For Ni concentrations up to 10% above the critical concentration $x_c \approx 0.026$ they found $T_C \propto (x-x_c)^{3/4}$ and a $T \ln T$ contribution to the specific heat down to 0.3 K. The $T$ dependence of the resistivity shows a power-law behavior

$$\rho(T \to 0) = \rho_0 + AT^n.$$  

(2.1)

The exponent $n$ displays a sharp minimum of $n = 5/3$ near $x_c$, while the prefactor $A$ shows an equally sharp maximum. These results are all consistent with the predictions of Hertz-Millis-Moriya theory; cf. Sec. III.C.2.

The lowest $T_C$ achieved in these experiments was $T_C \approx 7$ K at $x - x_c \approx 0.002$; see Fig. 14. This is on the same order as the temperature above which, e.g., MnSi displays behavior consistent with Hertz theory even though the behavior at low $T$ is very different. Subsequent ac susceptibility and zero-field-cooled and field-cooled magnetization measurements on the same samples at temperatures as low as 2 K found evidence for spin-glass freezing in a small region of the phase diagram $(0.025 \leq x \leq 0.028)$ (Nicklas, 2000). To corroborate this observation a measurement of the thermal expansion was performed on the same $x = 0.024$ polycrystal studied by Nicklas et al. (1999). Küchler et al. (2006) found that the Grüneisen ratio (i.e., the thermal expansion coefficient divided by the specific heat) does not increase with decreasing $T$, but remains constant below 3 K, in contrast to what is expected at a QCP (Küchler et al., 2006). Single-crystalline samples investigated by Franz et al. (2010) showed similar transport
TABLE III. Systems showing a second-order transition: Weakly disordered bulk systems. $T_C$ = Curie temperature, $\rho_0$ = residual resistivity, FM = ferromagnetism, SC = superconductivity, and n.a. = not available.

<table>
<thead>
<tr>
<th>System</th>
<th>Order of Transition$^a$</th>
<th>$T_C$/K</th>
<th>Magnetic moment/$\mu_0$$^b$</th>
<th>Tuning parameter</th>
<th>Disorder $(\rho_0/\mu_\Omega \text{cm})^c$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ni$<em>x$Pd$</em>{1-x}$</td>
<td>2nd$^d$</td>
<td>600 $- 7^e$</td>
<td>0.2 $- 2.45^f$</td>
<td>Composition$^1$</td>
<td>1.5 $\approx 7^f$</td>
<td>Low-T behavior unclear</td>
</tr>
<tr>
<td>(Ni$_x$Pd)$_3$Al</td>
<td>2nd$^g$</td>
<td>42 $- 4^h$</td>
<td>0.075 $- 0^i$</td>
<td>Composition$^3$</td>
<td>10$^i$</td>
<td>Moderate disorder</td>
</tr>
<tr>
<td>Ni$<em>3$Al$</em>{1-x}$Ga$_x$</td>
<td>2nd$^j$</td>
<td>41 $- 5^k$</td>
<td>0.075 $- 0.02^l$</td>
<td>Composition$^1$</td>
<td>n.a.</td>
<td>Disorder unknown</td>
</tr>
<tr>
<td>UIr</td>
<td>2nd$^m$</td>
<td>46 $- 1^n$</td>
<td>0.5$^o$</td>
<td>Pressure$^1$</td>
<td>n.a.$^f$</td>
<td>Three FM phases, coex. FM + SC</td>
</tr>
<tr>
<td>UNiSi$_2$</td>
<td>2nd$^p$</td>
<td>95.6$^q$</td>
<td>1.2$^r$</td>
<td>Pressure$^1$</td>
<td>$\approx 25$</td>
<td>Two FM phases</td>
</tr>
<tr>
<td>(Cr$_{1-x}$Fe)$_2$B</td>
<td>2nd$^s$</td>
<td>160 $- 0^t$</td>
<td>0.8$^u$</td>
<td>Composition</td>
<td>n.a.</td>
<td>Bulk powder samples</td>
</tr>
<tr>
<td>Sc$<em>2$Co$</em>{1-x}$P$_{1.2}$</td>
<td>2nd$^v$</td>
<td>35 $- 2^w$</td>
<td>0.1 $- 0.028^x$</td>
<td>Composition</td>
<td>n.a.</td>
<td>FM induced by dimer breaking</td>
</tr>
<tr>
<td>CeSi$_3$</td>
<td>2nd$^y$</td>
<td>9.5 $- 0^z$</td>
<td>0.2 $- 0.012$</td>
<td>Pressure</td>
<td>12 $\approx 30$</td>
<td>Moderate disorder</td>
</tr>
<tr>
<td>CePd$_{1-x}$Ni$_x$</td>
<td>2nd$^a$</td>
<td>10.5 $- 6.1^b$</td>
<td>n.a.</td>
<td>Composition</td>
<td>$\approx 15$</td>
<td>$T_C$ nonmonotonic</td>
</tr>
<tr>
<td>U$_2$Ru$_4$Ge$_6$</td>
<td>2nd$^c$</td>
<td>11.2 $- 3^d$</td>
<td>0.2$^e$</td>
<td>Pressure$^3$</td>
<td>58$^b$</td>
<td>Intermediate disorder</td>
</tr>
<tr>
<td>U$<em>4$(Ru$</em>{1-x}$Os$_x$)$_2$Ge$_6$</td>
<td>n.a.</td>
<td>12$^f$</td>
<td>0.2$^g$</td>
<td>Composition</td>
<td>n.a.</td>
<td>Disorder unknown</td>
</tr>
<tr>
<td>(Sc$_{1-x}$Lu$_x$)$_3$In</td>
<td>2nd$^h$</td>
<td>4 $- 1^i$</td>
<td>0.13 $- 0^j$</td>
<td>Composition</td>
<td>n.a.</td>
<td>Quasi-1-d chains of Sc-In</td>
</tr>
</tbody>
</table>

$^a$At the lowest temperature achieved.
$^b$Per formula unit unless otherwise noted.
$^c$For the highest-quality samples.
$^d$For $x = 1 - 0.027$ (Nicklas et al., 1999).
$^e$For 0.018 $\leq x \leq 0.1$ (Nicklas et al., 1999).
$^f$Nicklas (2000) and Tari and Coles (1971); Ikeda (1987) reported $\rho_0$ as small as 0.01 $\mu\Omega$ cm for the relevant Ni concentrations.
$^g$For $x = 0.9$.
$^h$For $x = 0$ (Niklowitz et al., 2005) to $x = 0.1$ (Sato, 1975).
$^i$For $x = 0.1$.
$^j$For $x = 0.33$.
$^k$RHR up to 250 (Kobayashi et al., 2006).
$^l$For $x = 0.05 - 0.02$.
$^m$For $x = 0-0.08$.
$^n$For $x = 0 - x \geq 0.7$.
$^o$Per Co for $x = 0.55 - 0.35$.
$^p$For $p = 0 - 13$ kbar.
$^q$At $T = 1.7$ K for $p = 0-14$ kbar.
$^r$For CeSi$_3$, at $p = 0$ with a current in the $a$ direction (Sato et al., 1988).
$^s$For $x = 0.5 - 0.94$.
$^t$For $p = 0-2$ GPa.
$^u$Per U.
$^v$For $x = 0.3$ (Colineau et al., 2001).
$^w$For $x = 0$.

and thermodynamic properties as those seen in polycrystals studied by Nicklas et al. (1999), but a detailed analysis of the magnetization indicates that at low fields and low temperatures the behavior is not consistent with either a mean-field QCP or a first-order transition. Considering that neutron-depolarization imaging experiments showed that polycrystalline samples are much more homogeneous than the single-crystal samples (Pfleiderer et al., 2010), these results raise the question of disorder present in the samples. The strength of the disorder, or how to characterize it, is not quite clear. The data obtained by Ikeda (1987) suggested a residual resistivity $\rho_0 \approx 5 \mu\Omega$ cm for $x$ around the critical concentration. Tari and Coles (1971) reported a low-temperature ($< 4.2$ K) resistivity of about $1 \mu\Omega$ cm for a sample with $x = 0.025$. $\rho_0$ for the samples studied by Nicklas (2000) is about $0.5 \mu\Omega$ cm for pure Pd (RRR = 40) and for $x = 0.1, 1.5 \mu\Omega$ cm for $x \approx x_c$, and it reaches a maximum of $5 \mu\Omega$ cm at $x \approx 0.04$. These results suggest that there is a substantial amount of disorder even in the best samples. It would be desirable to revisit the QPT in NiPd, while carefully characterizing the amount of disorder in the samples. A $\rho_0$ of $1 \mu\Omega$ cm would put the sample marginally in the intermediate regime II of the theory discussed in Sec. III.B.3, where the theory predicts a continuous transition with effectively mean-field exponents. However, if the spin-glass effects found by Nicklas (2000) were to be corroborated this theory would be inapplicable and the system would have to be classified with the materials discussed in Sec. II.E.

b. Ni$_3$Al$_{1-x}$Ga$_x$ and (Ni$_x$Pd)$_3$Al

The FM order in Ni$_3$Al with $T_C = 41$ K (see Sec. II.B.1.d) can be suppressed by substitution of Pd for Ni (Sato, 1975), or by doping with Ga (Yang et al., 2011). In the former system, a QCP is reached at $x \approx 0.095$, at which concentration the samples measured by Sato (1975) had a residual resistivity
### TABLE IV. Systems showing a second-order transition: strongly disordered bulk systems. $T_C$ = Curie temperature, $\rho_0 = \text{residual resistivity, FM = ferromagnetism, SC = superconductivity, and n.a. = not available.}$

<table>
<thead>
<tr>
<th>System</th>
<th>Order of transition</th>
<th>$T_C$/K</th>
<th>Magnetic moment/$\mu_B$</th>
<th>Tuning parameter</th>
<th>Disorder ($\rho_0/\mu\Omega$/cm)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>La$<em>{x}$Cr$</em>{1-x}$Ge$_3$</td>
<td>n.a.</td>
<td>55 – 20$^a$</td>
<td>1.4$^b$</td>
<td>comp + press$^d$</td>
<td>$100^b$</td>
<td>Lowest $T_C$ rather high</td>
</tr>
<tr>
<td>U$_2$Re$_2$,Si$_2$</td>
<td>2nd$^c$</td>
<td>25 – 2$^b$</td>
<td>0.4 – 0.03$^3$</td>
<td>Composition$^d$</td>
<td>$≈100^b$</td>
<td>Strong disorder</td>
</tr>
<tr>
<td>U$_2$Ru$_3$,Ge$_4$</td>
<td>2nd$^c$</td>
<td>$≈10 – 0^a$</td>
<td>$≈0.1 – 0^a$</td>
<td>Composition$^d$</td>
<td>n.a.$^i$</td>
<td>Disorder unclear</td>
</tr>
<tr>
<td>Th$_2$–$_5$Cu$_2$Si$_2$</td>
<td>2nd$^f$</td>
<td>101 – 12$^b$</td>
<td>0.92 – 0.09$^b$</td>
<td>Composition$^d$</td>
<td>$235^f$</td>
<td>Disorder unclear</td>
</tr>
<tr>
<td>UC$<em>{0.5}$Fe$</em>{0.5}$</td>
<td>2nd$^g$</td>
<td>8.5 – 3$^d$</td>
<td>0.1 – 0.02$^m$</td>
<td>Composition$^d$</td>
<td>$430^g$</td>
<td>Extremely high $\rho_0$</td>
</tr>
</tbody>
</table>

$^a$At the lowest temperature achieved.  
$^b$Per formula unit unless otherwise noted.  
$^c$For the highest-quality samples.  
$^d$For $x = 0.16$ and $p = 0 – 3$ GPa.  
$^e$For $x = 0$.  
$^f$For $x = 0.16$.  
$^g$For $x = 0.6 – 0.2$ (Butch and Maple, 2009).  
$^h$For $x = 0.1$ (Butch and Maple, 2010).  
$^i$For $x = 0–0.4$ (Huy, Gasparini, Klaasse et al., 2007).  
$^j$Large nominal $\rho_0$ $≈ 200 – 500 \mu\Omega$/cm due to cracks; not indicative of the intrinsic disorder (Huy, Gasparini, Klaasse et al., 2007).  
$^k$For $x = 1 – 0.15$.  
$^l$High $\rho_0$ due to microcracks.  
$^m$For $x = 0.75 – 0.22$.  
$^n$For $x = 0.22$.  
$^o$Lin et al. (2013).  
$^p$Bauer et al. (2005).  
$^q$Butch and Maple (2009).  
$^r$Huy, Gasparini, Klaasse et al. (2007).  
$^s$Lenkewitz et al. (1997).  
$^t$Huang et al. (2013).

### TABLE V. Systems showing a second-order transition: Quasi-one-dimensional (1D) materials. $T_C = \text{Curie temperature, } \rho_0 = \text{residual resistivity, and n.a. = not available.}$

<table>
<thead>
<tr>
<th>System</th>
<th>Order of Transition$^a$</th>
<th>$T_C$/K</th>
<th>Magnetic moment/$\mu_B$</th>
<th>Tuning parameter</th>
<th>Disorder ($\rho_0/\mu\Omega$/cm)$^d$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>YbNi$_2$P$_2$</td>
<td>2nd$^i$</td>
<td>0.14$^b$</td>
<td>$≈0.035^j$</td>
<td>None</td>
<td>$2.6^l$</td>
<td>Quasi-1D</td>
</tr>
<tr>
<td>YbNi$<em>4$(P$</em>{1-x}$As$_x$)$_2$</td>
<td>2nd$^i$</td>
<td>0.15 – 0.02$^k$</td>
<td>$≈0.05^l$</td>
<td>Composition$^d$</td>
<td>$5.5, 15^e,^3$</td>
<td>Quasi-1D, disordered</td>
</tr>
</tbody>
</table>

$^a$At the lowest temperature achieved.  
$^b$Per formula unit unless otherwise noted.  
$^c$For the highest-quality samples.  
$^d$For $x = 0–0.08$ (Steppke et al., 2013).  
$^e$5.5 for $J//c$, 15 for $J//c$ (Steppke et al., 2013).  
$^f$Kreller et al. (2011).  
$^g$Spehling et al. (2012).  
$^h$Steppke et al. (2013).  
$^i$$\rho_0 \approx 10 \mu\Omega$/cm, indicating moderate disorder.  
$^j$The observed critical behavior is consistent with the Hertz-Millis-Moriya theory, as one would theoretically expect for systems in this disorder regime; see Sec. III.B.3.  
$^k$As $\rho_0$ increases.  
$^l$With increasing pressure there are three distinct FM phases (Kotegawa et al., 2011).  
$^m$Ni$_3$Ga is paramagnetic (Hayden, Lonzarich, and Skriver, 1986).  
$^n$The disorder strength in these samples is not known, but assuming the same moderate disorder as in (Ni$_{1-x}$Pd$_x$)$_2$Al one would expect Hertz-type critical behavior according to the theoretical analysis reviewed in Sec. III.B.3.  
$^o$This is indeed borne out by the experiment (see Fig. 15): The temperature-concentration phase diagram obeys Eq. (3.51), the susceptibility at the critical concentration diverges as $T^{-4/3}$, Eq. (3.55), and the magnetization as a function of temperature near $T_C$ obeys Eq. (3.59). The first result reflects the combination of critical exponents $\nu/\sqrt{1 + 2\nu} = 3/4$; see Eq. (3.47).  
$^p$The second one reflects the exponent $\gamma_T = 4/3$, and if combined with the first one it also implies $\gamma = 1$, in agreement with Eq. (3.47), since $\gamma \sim T^{-4/3} \propto |x - x_c|^{-1}$.  
$^q$The third one reflects $\beta = 1/2$ in addition to the combination $\nu/\sqrt{1 + 2\nu}$. See Appendix B for the definitions of the critical exponents, and Sec. III.C.2.b for the scaling considerations underlying the above statements. As emphasized in Sec. III, this behavior is expected to hold, strictly speaking, only in a preasymptotic regime. However, for moderate disorder strengths the true asymptotic regime is expected to be unobservably small.  
$^r$UIr at ambient pressure is an Ising-like FM with $T_C \approx 46$ K. High-quality samples with a RRR $\approx 250$ have been investigated under hydrostatic pressure (Akazawa et al., 2004; Kobayashi et al., 2006, 2007). The overall phase diagram is similar to that of UGe$_2$, but the details are different.  
$^s$With increasing pressure there are three distinct FM phases labeled FM1, FM2, and FM3 (Kobayashi et al., 2006), and strain and resistivity measurements suggest that they have slightly different crystal structures (Kotegawa et al., 2011).
FM1 has an ordered moment of $0.5\mu_B/\text{U}$. There is a first-order metamagnetic transition between FM1 and FM2 at $p \approx 1.7$ GPa (at $T = 0$). The ordered moment in FM2 and FM3 is less than $0.5\mu_B/\text{U}$. $T_C$ goes to zero, and FM3 gives way to paramagnetism, at a critical pressure $p_c \approx 2.8$ GPa. Near $p_c$ in the FM3 phase, superconductivity is observed at $T < 140$ mK (Akazawa et al., 2004). The absence of metamagnetic behavior in the PM phase is indicative of the FM3-PM transition remaining second order to the lowest observed $T_C \approx 0.8$ K.

d. UNiSi$_2$

UNiSi$_2$ is a collinear ferromagnet with $T_C \approx 95$ K and U moments of $1.2\mu_B$ directed along the crystallographic $c$ axis of the orthorhombic structure (Geibel et al., 1990; Kaczorowski, 1996; Taniguchi et al., 1998; Das et al., 2000; Pikul, 2012). Although the amount of magnetic entropy below $T_C$, $\Delta S \approx 11.3$ J/mol K, suggests that the uranium moments are mostly localized, this value is lower than the value $\Delta S = R \ln(10) = 19.1$ J/mol K expected for fully localized $U^{3+}$ moments (Sidorov, Tobash et al., 2011). This is possibly due to the crystalline electric field and the Kondo effect, which is seen in the $\rho \propto -\ln T$ behavior of the resistivity above $T_C$ (Kaczorowski, 1996; Sidorov, Tobash et al., 2011). In polycrystalline samples as well as in single crystals of good quality ($\text{RRR} \approx 7$) the FM phase transition in zero field is second order, characterized by a $\lambda$-like peak in $C(T)/T$ (Pikul, 2012). Partial substitution of Th for U suppresses $T_C$ and leads to pronounced disorder effects; this system is discussed in Sec. II.E.2.a. Sidorov, Tobash et al. (2011) investigated single crystals of UNiSi$_2$ under hydrostatic pressure, up to about 6 GPa. With increasing pressure $T_C$ decreases, moderately for pressures up to about 4 GPa, and then more sharply, vanishing above 5.5 GPa; see Fig. 16. The FM phase transition remains second order (from ac-calorimetry measurements) in the pressure range $0 \leq p \leq 5.1$ GPa where the transition could be detected. In the pressure range near the QPT, between 5.1 and 5.5 GPa, another phase appears (small shaded region in Fig. 16), which is characterized by weak FM. This feature, which is reminiscent of the distinct FM phases in UIr (see Sec. II.C.1.c), is indicated by an upturn in the ac susceptibility and signatures in the resistivity and the specific heat (Sidorov, Tobash et al., 2011). The magnetic entropy is strongly reduced under pressure. This and the enhanced Sommerfeld coefficient and resistivity at 5.5 GPa led Sidorov, Tobash et al. (2011) to suggest a delocalization of the $f$ electrons at the QPT.

e. $(\text{Cr}_{1-x}\text{Fe})_2\text{B}$

Ferromagnetism can be induced in the paramagnetic compound Cr$_2$B by doping with Fe. The resulting system $(\text{Cr}_{1-x}\text{Fe})_2\text{B}$ undergoes a QPT near $x_c = 0.02$ (Schoop et al., 2014). Doping introduces a substantial amount of disorder resulting in a residual resistivity near $x_c$ of $\rho_0 \approx 35 \mu\Omega\text{cm}$. The exponent $n$ in Eq. (2.1) falls from its Fermi-liquid value $n = 2$ at $x = 0$ to $n = 1$ at $x_c$, and remains there for larger values of $x$. The prefactor $A$ peaks around $x_c$. However, the absolute change of $\rho$ with temperature in the linear-in-$T$ range is extremely small. For instance, for $x = 0.025$ between 8 and 20 K $\Delta \rho = \rho - \rho_0 \approx 0.2 \mu\Omega\text{cm}$, which is very small compared to the rather large value of $\rho_0 = 40 \mu\Omega\text{cm}$. This results in a small value of $A \approx 18 \text{n}\Omega\text{cm}/K$; for other values of $x$ it is even smaller. A magnetic field of 14 T restores the Fermi-liquid value $n = 2$. The specific heat shows a $T \ln T$ term similar to that observed in Ni$_{1-x}$Pd$_x$, with a prefactor that is largest around $x_c$, but again the maximal change is very small, $\Delta C/T \approx 2 \text{mJ/K}^2\text{mol}$. These observations are in principle consistent with the existence of a QCP at $x_c$, but the NFL properties characteristic of critical behavior are extremely weak. Given the disorder strength, the theory reviewed in Sec. III.B predicts a continuous transition.

f. $\text{Zr}_{1-x}\text{Nb},\text{Zn}_2$

Itinerant ferromagnetism in $\text{ZrZn}_2$ ($T_C = 28.5$ K) (Matthias and Bozorth, 1958; Pickart et al., 1964) can be tuned to zero by substituting Nb for Zr. Sokolov et al. (2006) investigated the magnetization of several polycrystalline samples of the series $\text{Zr}_{1-x}\text{Nb},\text{Zn}_2$ with $0 \leq x \leq 0.14$ down to 1.8 K. From an Arrrott-plot analysis they found that $T_C$ is suppressed to zero at $x_c \approx 0.08$. The $\rho(x)$ dependence of $T_C$ is well described by $T_C \propto (x-x_c)^{3/4}$, and the spontaneous moment vanishes linearly with $T_C$. Furthermore, the inverse magnetic susceptibility could be fitted to $\chi^{-1} = aT^{4/3} + b(x-x_c)$, with $a$ and $b$ constants. All of this is consistent with the results of Hertz-Millis-Moriya theory; see Sec. III.C.2.b. There were no indications of a first-order transition or metamagnetism.

These results are reminiscent of those for Ni$_{1-x}$Pd$_x$ (cf. Sec. II.C.1.a). However, in the present case little is known about the disorder present in the samples. X-ray diffraction experiments revealed single-phase specimens with Laves phase C15 structure, but no resistivity data are available (Sokolov, 2015).

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\textsuperscript{14}Pikul (2012) found an even lower value, $\Delta S \approx 8$ J/mol K.
showed that there is indeed a QCP.

Yoshimura (2008) studied powder samples and concluded that at $x ≈ 0.7$ (Kanbayasi, 1978; Kiyama et al., 1999), which led to the proposal of a QCP in this material. Itoh, Mizoguchi, and Yoshimura (2008) studied powder samples and concluded from magnetization measurements that there is indeed a QCP. From Arrrott plots for $x$ near $x_c$, they inferred a field dependence of the magnetization $M$ and specific heat and concluded that at $x = x_c$ they inferred a field dependence of the magnetization $M$ and specific heat and found $\delta = 1.6$ in agreement with Itoh, Mizoguchi, and Yoshimura (2008). Their scaling analysis yields a very unusual temperature dependence of the specific-heat coefficient $\gamma \propto T^{0.7}$. The behavior is markedly different from that of ceramic samples (see Sec. II.B.4.a) and epitaxial thin films (see Sec. II.E.3).

g. Sr$_{1-x}$Ca$_x$RuO$_3$ (bulk powder samples)

Early studies of polycrystalline Sr$_{1-x}$Ca$_x$RuO$_3$ samples showed that $T_C$ goes to zero linearly as $x$ approaches $x_c \approx 0.7$ (Kanbayasi, 1978; Kiyama et al., 1999), which led to the proposal of a QCP in this material. Itoh, Mizoguchi, and Yoshimura (2008) studied powder samples and concluded from magnetization measurements that there is indeed a QCP. From Arrrott plots for $x$ near $x_c$, they inferred a field dependence of the magnetization $M$ and specific heat and found $\delta = 3/2$. This agrees with the prediction of the generalized Landau theory for disordered systems described in Sec. III.B.3. No information is available about the disorder strength in these samples. Huang et al. (2015) studied the dynamical scaling of the magnetization and specific heat and found $\delta = 1.6$ in agreement with Itoh, Mizoguchi, and Yoshimura (2008). Their scaling analysis yields a very unusual temperature dependence of the specific-heat coefficient $\gamma \propto T^{0.7}$. The behavior is markedly different from that of ceramic samples (see Sec. II.B.4.a) and epitaxial thin films (see Sec. II.E.3).

h. SrCo$_2$(Ge$_{1-x}$P)$_x$

For the ferromagnetism that develops in SrCo$_2$(Ge$_{1-x}$P)$_x$ at $x \approx 0.325$, Jia et al. (2011) proposed a new tuning mechanism: the breaking of bonds in Ge-Ge dimers, in which they argue is more important than the simple change in carrier concentration with $x$. SrCo$_2$Ge$_2$ forms in the ThCr$_2$Si$_2$ structure, with Co$_2$Ge$_2$ layers separated by a Ge-Ge interlayer bond, i.e., a dimer. This causes the layers to be pulled apart and form a collapsed tetragonal cell with a three-dimensional electronic structure. SrCo$_2$Ge$_2$ is a simple Pauli paramagnet. The lack of such a dimer in SrCo$_2$P$_2$ causes the same structure to be uncollapsed and, therefore, to have a more two-dimensional electronic structure and a shorter Co-Co separation within the layers, which increases the Co-Co interaction. From measurements of the magnetization, the specific heat, and the specific heat of polycrystalline samples Jia et al. (2011) concluded that at $x \approx 0.325$ the system develops bulk ferromagnetism via a QCP. The Curie temperature increases with increasing $x$, reaches a maximum of $T_C \approx 35$ K at $x \approx 0.55$, and then decreases. For $x \geq 0.75$ Jia et al. (2011) found that the ground state is a Stoner-enhanced paramagnet rather than a ferromagnet. The ferromagnetism is of the itinerant type, characterized by a Curie-Weiss behavior with an effective moment much larger than the saturation moment. This is in agreement with band-structure calculations, which show a maximum in the density of states at $x \approx 0.5$, where $T_C$ reaches its maximal value (Cuervo-Reyes and Nesper, 2014). $T_C$ increases linearly from $x = 0.3$ to 0.5 having a value of about 5 K for $x = 0.35$. No sign of a first-order transition or spin-glass behavior was detected. For a sample with $x = 0.325$ and no $T_C$ the susceptibility was found to behave as $\chi \propto T^{-4/3}$ (down to 2 K) and the specific heat $C/T \propto -\ln T$ (down to 0.4 K), in agreement with Hertz-Millis-Moriya theory (cf. Sec. III.C.2).

No resistivity measurements have been reported, and the role of disorder in this material is unknown.

i. CeSi$_3$

CeSi$_3$ can be considered a FM dense Kondo system (Yashima et al., 1982; Sato et al., 1988). It crystallizes in...
the $\alpha$-ThSi$_2$ structure with a broad homogeneity range $1.7 \leq x \leq 2$ (Ruggiero and Olcese, 1964; Yashima et al., 1982; Yashima and Satoh, 1982). It shows a paramagnetic ground state for $x \geq 1.85$, while a magnetically ordered state was found for $x \leq 1.8$ with a transition temperature around 10 K. The nature of the magnetic order is not clear (Drotziger et al., 2006). Susceptibility measurements suggest that the magnetic structure depends on the Si vacancy distribution (Shaheen and Mendoza, 1999), and magnetization measurements indicate that the ground state may not be pure FM, but rather a ferromagnet or canted ferromagnet resulting from the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction between the Ce 4f local moments on two different lattice sites (Drotziger et al., 2006).

Drotziger et al. (2006) studied the magnetization of a single crystal of CeSi$_{1.81}$ as a function of temperature and hydrostatic pressure. Pressure was found to suppress $T_C$ monotonically from its ambient-pressure value of 9.5 K. $T_C$ vanishes at a critical pressure $p_c \approx 13.1$ kbar, with a concomitant continuous suppression of the magnetic moment from 0.2$\mu_B$/Ce to zero. The transition at $T_C$ was found to be of second order down to the lowest observed value of $T_C \approx 3$ K. The electrical resistivity of the $x = 1.81$ sample is not known, but for $x = 1.86$ a residual resistivity of 12 $\mu$Ω cm (with current along the $a$ axis) and 30 $\mu$Ω cm (with current along the $c$ axis) has been reported (Sato et al., 1988). If these values are representative for the $x = 1.81$ critical sample as well, they put the system CeSi$_x$ in a moderately disordered regime where a continuous transition is expected theoretically; see Sec. III.B.3. However, questions about both the nature of the ordered phase and the nature of the transition at low $T$ remain (Drotziger et al., 2006).

j. CePd$_{1-x}$Ni$_x$

The FM $T_C$ of CePd ($T_C = 6.5$ K) initially increases upon alloying with nickel, then decreases for $x \geq 0.8$, and vanishes at a Ni concentration $x_c \approx 0.95$ (Kappler et al., 1997). Measurements of the specific heat, magnetization, and resistivity have shown NFL behavior of the resistivity for $3 < T < 30$ K, and logarithmic behavior of the specific-heat coefficient in a temperature window between about 1 and 10 K.

k. (Sc$_{1-x}$Lu$_x$)$_3$In

Upon doping of the nonstoichiometric FM compound Sc$_3$In with lutetium, evidence for a QCP with unusual values of the critical exponents in (Sc$_{1-x}$Lu$_x$)$_3$In has been found near a critical concentration $x_c \approx 0.035$ (Svanidze et al., 2015). NFL behavior has been observed in both the FM and PM phases, in addition to the vicinity of the QCP. This material may be characterized by a reduced dimensionality due to the one-dimensional nature of the Sc-In chains (Jeong and Kwon, 2007; Svanidze et al., 2015).

l. U$_4$Ru$_2$Ge$_6$ and U$_4$(Ru$_{1-x}$Os$_x$)$_7$Ge$_6$

U$_4$Ru$_2$Ge$_6$ is ferromagnetic below $T_C \approx 12$ K; it is a metal with Kondo-like and heavy-fermion features, while U$_4$Os$_x$Ge$_6$ is a paramagnet. The system U$_4$(Ru$_{1-x}$Os$_x$)$_7$Ge$_6$ has been investigated by Colineau et al. (2001), who found that $T_C$ is suppressed to zero for $x \approx 0.3$.

Hydrostatic pressure applied to U$_4$Ru$_2$Ge$_6$ also suppresses $T_C$, with a FM-PM QPT at $p \approx 2.6$ GPa. Resistivity measurements on polycrystalline samples by Hidaka et al. (2011) suggest the existence of a QCP with Hertz-type behavior. The residual resistivity at $p = 2.36$ GPa was about 58 $\mu$Ω cm. The residual resistivity of the U$_4$(Ru$_{1-x}$Os$_x$)$_7$Ge$_6$ samples is not known, but is presumably higher. This places this system in the disordered regime II (intermediate disorder) discussed in Sec. III.B.3, which is consistent with the observations.

2. Strongly disordered systems

a. LaV$_{1-x}$Cr$_x$Ge$_3$

Upon substitution of vanadium for chromium in LaCrGe$_3$, the FM $T_C$ drops from 88 to 36 K for $x = 0.21$ (Lin et al., 2013). Pressure applied to a sample with $x = 0.16$ leads to a further decrease of $T_C$, with a QPT at $p \approx 3$ GPa. The lowest $T_C$ achieved was about 20 K; the order of the QPT is not known.

b. URu$_{2−\gamma}$Re$_x$Si$_2$

The parent compound of URu$_{2−\gamma}$Re$_x$Si$_2$, URu$_2$Si$_2$, is a heavy-fermion superconductor (superconducting $T_c \approx 1.5$ K) that has an ordered phase of unknown nature, usually referred to as the “hidden-order” phase, below about 17 K; see Mydosh and Oppeneer (2011) for a recent overview. Substitution of Re, Tc, or Mn leads to the destruction of the hidden-order phase and the emergence of ferromagnetism past a certain dopant concentration (Dalichaouch et al., 1990). Only URu$_{2−\gamma}$Re$_x$Si$_2$ has been studied in detail. In this system, the hidden-order phase disappears for $x \approx 0.1$ and the system develops a FM ground state for $x \gtrsim 0.15$ (Butch and Maple, 2010). $T_C$ increases monotonically with increasing $x$ and reaches a maximum of almost 40 K at $x \approx 0.8$, above which the material does not remain in a single phase. The existence of FM long-range order has been ascertained by neutron scattering for $x = 0.8$ (Torkovich et al., 1992) and by $^{29}$Si NMR for $x \gtrsim 0.4$ (Kohori et al., 1993). Pronounced NFL behavior has been observed in the specific heat and the electrical resistivity for a large concentration range $0.15 \lesssim x \lesssim 0.8$ (Bauer et al., 2005), and the dynamical magnetic susceptibility shows unusual behavior for $0.2 < x < 0.6$ (Krishnamurthy et al., 2008). The system is highly disordered as judged from the residual resistivity, which is on the order of 100 $\mu$Ω cm (Butch and Maple, 2010), and the magnetic moment appears to go to zero continuously. Attempts to determine critical exponents have been hampered by difficulties in precisely determining the critical concentration (Bauer et al., 2005; Butch and Maple, 2009, 2010). Results from scaling plots yield exponents $\delta$ and $\gamma$ that vary continuously with $x$ and approach 1 and 0, respectively, for $x$ approaching the critical value $x_c \approx 0.15$, while $\beta \approx 0.8$ is independent of $x$ (Butch and Maple, 2009, 2010). These results are hard to understand within any phase-transition scenario, even if one interprets the exponents as effective ones in a preasymptotic region. $\gamma = 0$, in particular, contradicts the very notion of a FM order parameter. There currently is no resolution of this problem. The uncertainty about $x_c$ may be to
blame, and the strong disorder may lead to unusual effects. For instance, it is conceivable that there is a Re concentration region that represents a quantum Griffiths phase (see Secs. II.E and III.D.1) rather than true long-range FM order. It has also been speculated that an interplay between remnants of the hidden order and ferromagnetism leads to unusual behavior near the onset of ferromagnetism (Butch and Maple, 2010).

c. URh$_{1-x}$Ru$_x$Ge

Doping URhGe with Ru decreases $T_C$ after a small initial increase and suppresses it to zero at a Ru concentration close to $x = 0.38$ (Huy, Gasparini, Klaasse et al., 2007; Sakarya et al., 2008); see Fig. 8. Huy, Gasparini, Klaasse et al. (2007) found the QPT to be second order with a pronounced $T$ in $T$ contribution to the specific heat at the critical concentration. The $T$ dependence of the electrical resistivity shows a non-Fermi-liquid $T^n$ behavior, with $n < 2$ over a wide range of concentrations, with a minimum of $n = 1.2$ at the critical concentration. Such NFL behavior has been interpreted as indicative of the existence of a QPT. The continuous nature of the transition is consistent with theoretical expectations, assuming that the large critical Ru concentration leads to a substantial amount of disorder. The strength of the microscopic disorder is hard to determine experimentally, since cracks in the brittle system lead to an artificially high residual resistivity of 200–300 $\mu\Omega$ cm. The Grüneisen parameter $\Gamma$ is observed to stay finite at the transition, in disagreement with a theoretical result that predicts a diverging $\Gamma$ (Zhu et al., 2003).

Si doping up to $x \approx 0.2$ has little effect on the $T_C$ of URhGe$_{1-x}$Si$_x$ (Fig. 8), and no quantum phase transition has been observed in this material (Sakarya et al., 2008).

d. UCo$_{1-x}$Fe$_x$Ge

Doping of the weak FM UCoGe (see Sec. II.B.2.c) with Fe initially increases $T_C$ to a maximum of $T_C \approx 8.5$ K around $x = 0.075$. With further increasing $x$, $T_C$ decreases and vanishes at an extrapolated $x_c = 0.22$ (Huang et al., 2013). The QPT is believed to be second order, and there is some evidence for quantum-critical behavior in the transport and specific-heat data. Since the transition in UCoGe is first order, this implies the existence of a tricritical point in the phase diagram, but this has not been investigated. The origin of the very large residual resistivity is not clear.

e. Th$_{1-x}$UCu$_2$Si$_2$

UCu$_2$Si$_2$ orders ferromagnetically below $T_C \approx 101$ K, and ThCu$_2$Si$_2$ is paramagnetic. In Th$_{1-x}$UCu$_2$Si$_2$, Lenkewitz et al. (1997) found a QPT for $x \approx 0.15$, with a logarithmic $T$ dependence of the specific-heat coefficient. The large residual resistivity ($\rho_0 > 200$ $\mu\Omega$ cm) is due to microcracks in the samples and not a measure of intrinsic disorder.

3. Quasi-one-dimensional systems

a. YbNi$_4$P$_2$

YbNi$_4$P$_2$ is the stoichiometric metallic ferromagnet with the lowest $T_C$ observed to date, $T_C = 0.15$ K (Steppke et al., 2013). In this compound Ni is not magnetic (Députier et al., 1997; Krellner et al., 2011). The Yb atoms are arranged in chains along the $c$ direction and are located between Ni tetrahedra, forming a $\text{ZrFe}_2\text{Si}_2$ structure type with a lattice-constant ratio $c/a \approx 0.5$. Band-structure calculations show quasi-1D Fermi surfaces (Krellner et al., 2011), which are believed to be responsible for the observed anisotropy of the resistivity, $\rho_\parallel/\rho_\perp \approx 5$ at 1.8 K (Krellner and Geibel, 2012).

The Yb$^{3+}$ ion is located in an orthorhombic crystalline electric field (CEF) which splits the $J = 7/2$ energy levels, leaving a Kramers doublet as the ground state (Huesges et al., 2013), and causes the crystalline $c$ axis to be the magnetic easy axis (Krellner and Geibel, 2012). This can be seen in Fig. 17, which shows the ac susceptibility $\chi'(T)$ in a small field $H$ parallel and perpendicular, respectively, to the $c$ axis. Although YbNi$_4$P$_2$ is a heavy-fermion system with a Kondo temperature of 8 K, a small unscreened moment of about 0.05 $\mu_B$/Yb orders ferromagnetically at 0.15 K (Krellner et al., 2011; Spehling et al., 2012; Steppke et al., 2013; Gegenwart et al., 2015). In addition, despite the strong CEF anisotropy the moments align within the $ab$ plane, i.e., the magnetically hard direction (see Fig. 17). YbNi$_4$P$_2$ shares this uncommon behavior with just a few other FM Kondo-lattice systems, such as CeRuPO (Sec. II.D.2.a), CeAgSb$_2$ (Sec. II.D.3.a), YbNiSn (Bonville et al., 1992), and Yb(Rh$_{0.75}$Co$_{0.25}$)$_2$Si$_2$ (Sec. II.D.3.c). An explanation within a local-moment Heisenberg model with competing exchange interactions (Andrade et al., 2014) does not work for YbNi$_4$P$_2$ where quantum effects are strong. For instance, $\chi'(T)$ in classical Ising or Heisenberg systems is characterized by a power-law behavior at the transition with well-known universal exponents, while the divergence of $\chi'_\perp$ in Fig. 17 just above $T_C$ is much stronger than a power law. It is even stronger than what is expected for a pure

![FIG. 17. Temperature dependence of the ac susceptibility $\chi'(T)$ measured with a modulation-field amplitude $\mu_0H = 15$ $\mu$T parallel and perpendicular, respectively, to the $c$ axis. The large value $\chi'_\parallel(T_C) \approx 200 \times 10^{-6}$ m$^3$/mol indicates a FM phase transition. The dashed line indicates that $\chi'_\parallel(T) \propto T^{-2/3}$ above $T_C$. From Steppke et al., 2013.](image-url)
one-dimensional Ising ferromagnet, where \( \chi'(T) \propto T^{-1}\exp(2J/k_BT) \), with \( J \) the coupling constant (Ising, 1925). Another proposed explanation is strong transverse spin fluctuations which at sufficiently low temperature dominate the magnetic anisotropy (Krüger, Pedder, and Green, 2014). This model, however, implies a first-order phase transition at \( T_C \), which is not observed in YbNi\(_4\)P\(_2\) (Steppke et al., 2013).

YbNi\(_4\)P\(_2\) also has another of no other unconventional properties. For instance, polycrystals in a broad \( T \) region above \( T_C \) show strong NFL behavior in the resistivity \( \rho(T) \propto T \), the specific heat \( C/T \propto T^{-0.42} \), and the NMR relaxation rate \( 1/T_1T \propto T^{-3/4} \) (Krellner et al., 2011; Sarkar et al., 2012).

\( b. \) YbNi\(_4\)(P\(_{1-x}\)As\(_x\))\(_2\)

The results reviewed in Sec. II.C.3.a motivated the growth of single crystals with phosphorus substituted by arsenic (which amounts to negative chemical pressure) in order to reduce \( T_C \) and look for FM quantum criticality. Four single crystals of the series YbNi\(_4\)(P\(_{1-x}\)As\(_x\))\(_2\) were grown with a minimum value of \( \rho_0 = 5.5 \mu\Omega \cdot \text{cm} \) for the stoichiometric YbNi\(_4\)P\(_2\). Steppke et al. (2013) investigated the magnetic and thermodynamic properties down to 20 mK and, in particular, measured the Grüneisen ratio \( \Gamma(T) = \beta(T)/C(T) \), where \( \beta(T) \) is the volume thermal expansion coefficient. According to Zhu et al. (2003), this quantity should diverge as \( \Gamma(T) \propto T^{-\lambda} \) at any QCP, where \( \lambda = 1/(\nu z) \) is given in terms of the correlation-length exponent \( \nu \) and the dynamical exponent \( z \) (cf. Sec. III). Steppke et al. (2013) found that the FM phase transition is suppressed at \( x_c \approx 0.1 \) [Fig. 18(a)] and that it remains second order even in the sample with \( x = 0.08 \) with a \( T_C \) of about 25 mK. Both \( C(T)/T \) and \( \beta(T)/T \) diverge [see Fig. 18(b)] with exponents that are approximately independent of the As concentration, which rules out a possible quantum Griffiths phase (cf. Sec. III.D.1). Most importantly, \( \Gamma(T) \propto T^{-0.22} \) in the sample with \( x = 0.08 \) which is located almost at \( x_c \) [see Fig. 18(b)]. This provides evidence that in YbNi\(_4\)(P\(_{1-x}\)As\(_x\))\(_2\) a FM QCP exists. The nature of this QCP is still unclear. The exponent \( \lambda = 0.22 \) yields a value of \( \nu z \approx 5 \), which is rather large. For instance, within Hertz-Millis-Moriya theory one has \( \nu = 1/2, z = 3 \) (cf. Sec. III.C.2.b). This is not surprising, as no existing theory is expected to apply to this material. Any theoretical framework will have to take into account the local nature of the Yb 4f states with spin-orbit coupling, a strong Kondo effect, and the quasi-1D electronic structure. The absence of a first-order transition is likely due to the latter; see the following discussion.

4. Discussion, and comparison with theory

The materials in which a continuous transition is observed to the lowest temperatures achieved have been grouped into three distinct classes: see Tables III, IV, and V: weakly disordered, strongly disordered, and quasi-one-dimensional. For the last group, the conduction-electron system is expected to be a Fermi liquid (FL) at asymptotically low temperatures, but it will cross over to a Luttinger liquid at a temperature that depends on the electronic anisotropy. A determination of the temperature range where the theories discussed in Sec. III, which all depend on an underlying Fermi liquid, still apply requires detailed theoretical considerations that are currently not available. For the strongly disordered bulk systems, the theory discussed in Sec. III.C.3 predicts a continuous transition, and Griffiths effects may also be present; see Sec. III.D.1. Of the two systems in this category, URu\(_2\)Si\(_2\) and CeSi\(_2\) are the more thoroughly studied one. As discussed in Sec. II.C.2.b, the current experimental results cannot be easily interpreted with any existing theory. A major obstacle is the uncertainty about the critical concentration \( x_c \), and additional studies about the onset of long-range FM order are desirable.

In the weakly disordered group, CeSi\(_2\) and Ni\(_3\)Pd\(_{1-x}\) come with open questions regarding the nature of the transition, or the presence of phases other than the FM one, at low temperatures. Given the residual resistivities of these materials, theoretically one would expect a continuous transition in CeSi\(_2\), and a first-order one in Ni\(_3\)Pd\(_{1-x}\), provided the transition is not preempted by a different phase. In (Cr\(_{1-x}\)Fe\(_x\))\(_2\)B one expects a second-order transition, and the observed mean-field critical behavior is consistent with theoretical expectations; see Sec. III.B.3. For the remaining systems no information about the disorder strength is available, which makes a comparison with theoretical predictions difficult.

These somewhat inconclusive results may well have to be revisited if cleaner samples and/or measurements at lower temperatures should become available in the future. The history of ZrZn\(_2\), Sec. II.B.1.b, shows that improving sample quality can change the conclusion about the order of the transition. One also needs to keep in mind that the tricritical temperatures listed in Tables I and II span a substantial range,
TABLE VI. Systems showing change into a spin-density-wave (SDW) or antiferromagnetic (AFM) order. $T_C$ = Curie temperature, $T_N$ = Néel temperature, $\rho_0$ = residual resistivity, QC = quantum critical, and n.a. = not available.

<table>
<thead>
<tr>
<th>System</th>
<th>Order of transition</th>
<th>$T_C$/K</th>
<th>$T_N$/K</th>
<th>Magnetic moment/$\mu_0$</th>
<th>Tuning parameter</th>
<th>Disorder $(\rho_0/\rho\Omega cm)^{-1}$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nb$<em>{1-x}$Fe$</em>{2+y}$</td>
<td>1st (?$^{1,1}$)</td>
<td>72 $-$ 6$^p$</td>
<td>32 $-$ 2.8$^p$</td>
<td>$\approx 0.02^2$</td>
<td>Composition$^3$ pressure$^7$</td>
<td>5.5$-$17$^{1,14}$</td>
<td>SDW phase, Lifshitz point</td>
</tr>
<tr>
<td>PrPtAl</td>
<td>1st$^{6,6}$</td>
<td>4.7$^q$</td>
<td>5.85, 5.5$^{b,6}$</td>
<td>$\approx 1^6$</td>
<td>None</td>
<td>n.a.</td>
<td>Spiral phase, two SDW phases</td>
</tr>
<tr>
<td>CeFe$_{1-x}$P$_x$O</td>
<td>n.a.</td>
<td>$\approx 10$ $-$ 6$^{9,10}$</td>
<td>$\approx 8^{m,11}$</td>
<td>0.95 $-$ 0$^q$</td>
<td>Composition$^{9,10}$</td>
<td>n.a.</td>
<td>Conflicting results</td>
</tr>
<tr>
<td>CeRu$_{1-x}$Fe$_x$PO</td>
<td>n.a.</td>
<td>15 $-$ 0.3$^{12}$</td>
<td>$\approx 0.5$ $-$ 0$^{12}$</td>
<td>1.2$^{13}$</td>
<td>Composition$^{12}$</td>
<td>n.a.</td>
<td>Q.C fluctuations, possible AFM phase</td>
</tr>
<tr>
<td>CeAgSb$_3$</td>
<td>1st$^{14}$</td>
<td>9.6 $-$ 2$^{14}$</td>
<td>$\approx 6$ $-$ 4$^{14}$</td>
<td>0.41$^{15}$</td>
<td>Pressure$^{14}$ $\downarrow$ field$^{16}$</td>
<td>0.2$^{14}$</td>
<td>AFM phase not always observed</td>
</tr>
<tr>
<td>CeRu$<em>2$(Ge$</em>{1-x}$Si$_x$)$_2$</td>
<td>n.a.</td>
<td>8 $-$ 2.5$^{17}$</td>
<td>10 $-$ 1$^{17}$</td>
<td>n.a.</td>
<td>Comp./press.$^{18}$</td>
<td>0.3$^{18}$</td>
<td>Hybridization suppresses FM</td>
</tr>
<tr>
<td>Yb(Rh$<em>{0.73}$Co$</em>{0.27}$)$_2$Si$_2$</td>
<td>n.a.</td>
<td>1.3$^{n,19}$</td>
<td>1.2 $-$ 0.07$^{m,20}$</td>
<td>0.1 $-$ 0.002$^{b,22,19}$</td>
<td>Composition$^{20}$ pressure$^{23,24}$</td>
<td>0.5$-$10$^{20,21}$</td>
<td>Field-induced AFM QPT</td>
</tr>
</tbody>
</table>

$^a$For the FM-to-AFM or SDW transition at the lowest temperature achieved.
$^b$Per formula unit unless otherwise noted.
$^c$For the highest-quality samples.
$^d$For the FM-SDW transition. The FM-PM transition on the Nb-rich side is second order to the lowest $T_C$ measured ($\approx 2$ K).
$^e$For $0.04 < y < 0.007$ (Brando et al., 2008; Moroni-Klementowicz et al., 2009).
$^f$For $0.015 > y > -0.01$ (Brando et al., 2008; Moroni-Klementowicz et al., 2009).
$^g$For the FM-SDW2 and SDW2-SDW1 transitions. The SDW1-PM is 2nd order (Abdul-Jabbar et al., 2015).
$^h$For the SDW1-PM and SDW2-SDW1 transitions, respectively (Abdul-Jabbar et al., 2015).
$^i$For the FM-AFM transition at $T < 9$ K. The order of the transition at low $T$ is not known.
$^j$For $0 < p < 0.7$ GPa and AFM for $0.7 < p < 2.8$ GPa (Kotegawa et al., 2013; Lengyel et al., 2015).
$^k$For the FM-AFM transition.
$^l$For $x = 0.4$ $-$ 0.8 (Luo et al., 2010; Jesche et al., 2012).
$^m$For $x = 0.9$ (Jesche, 2011).
$^n$For Yb(Rh$_{0.73}$Co$_{0.27}$)$_2$Si$_2$ (Lausberg et al., 2013).
$^o$For $x = 0.27$ $-$ 0 (Klingner et al., 2011).
$^p$0.002 $\mu_B$ for YbRh$_2$Si$_2$ (Ishida et al., 2003); 0.1 $\mu_B$ for Yb(Rh$_{0.73}$Co$_{0.27}$)$_2$Si$_2$ (Lausberg et al., 2013).
$^q$Friedemann (2015).
$^r$Brando et al. (2008).
$^s$Moroni-Klementowicz et al. (2009).
$^t$Friedemann et al. (2013).
$^u$Duncan et al. (2010).
$^w$Kotegawa et al. (2013).
$^x$Lengyel et al. (2015).
$^y$Luo et al. (2010).
$^z$Jesche et al. (2012).
$^\alpha$Jesche (2011).
$^\beta$Kitagawa et al. (2012).
$^\gamma$Krellner et al. (2007).
$^\delta$Sidorov et al. (2003).
$^\epsilon$Araki et al. (2003).
$^\zeta$Logg et al. (2013).
$^\iota$Süllow et al. (1999).
$^\kappa$Wilhelm and Jaccard (1998).
$^\lambda$Lausberg et al. (2013).
$^\mu$Klingner et al. (2011).
$^\nu$Krellner et al. (2009).
$^\xi$Ishida et al. (2003).
$^\omega$Mederle et al. (2001).
$^\theta$Knebel et al. (2006).

and the $T_K$ in, for instance, URhGe is barely higher than the lowest temperature at which UIJ measurements were performed.

D. Systems changing to spin-density-wave or antiferromagnetic order

In some systems the FM phase undergoes a transition to a spatially modulated magnetic phase as $T_C$ decreases; see the schematic phase diagram in Fig. 2(c).15 This produces a Lifshitz point, where the FM, modulated, and PM phases meet, as well as two QPTs, one from the FM phase to the modulated one, and one from the modulated phase to the PM. They are discussed later, and their properties are summarized in Table VI. In some of these materials the evidence for a modulated phase is stronger than in others, and in some cases there are conflicting experimental results. The classification of some of these systems within our scheme should thus be considered tentative.

1. Simple ferromagnets

a. An itinerant magnet: Nb$_{1-y}$Fe$_{2+y}$

The Laves phase compound NbFe$_2$ shows itinerant antiferromagnetism on the border of a FM phase. First indications of a low-$T$ AFM ordered state with $T_N \approx 10$ K were found in magnetization and NMR experiments on polycrystals (Shiga and Nakamura, 1987; Yamada and Sakata, 1988; Crook and Cywinski, 1995). This was recently confirmed by a microscopic study with electron spin resonance, muon-spin relaxation, and Mössbauer spectroscopy on single crystals (Rauch et al., 2015). However, this state is characterized by an unusually high magnetic susceptibility, $\chi \approx 0.02$ in SI units, which corresponds to a large Stoner enhancement factor of the order of 180 (Brando et al., 2008). They speculated that the magnetic order in NbFe$_2$ is a long-wavelength modulated state with a small ordering wave number $Q \approx 0.05$ Å$^{-1}$. This
was corroborated by recent neutron scattering experiments (Niklowitz, 2015).

The Wilson ratio (i.e., the low-\(T\) susceptibility divided by the specific-heat coefficient) is about 60, which indicates that the susceptibility is much more enhanced than the effective electron mass. This suggests that stoichiometric NbFe\(_2\) is very close to the border of ferromagnetism. Indeed, a FM ground state is found in iron-rich samples (Yamada and Sakata, 1988; Crook and Cywinski, 1995; Moroni-Klementowicz et al., 2009). By varying the iron content in Nb\(_{1-y}\)Fe\(_{2+y}\) within a narrow homogeneity range, NbFe\(_2\) can be tuned from ferromagnetism for \(y \geq 0.01\) via an intermediate SDW modulated state around \(y = 0\) to a QCP at \(y \approx 0.015\). For \(y < -0.015\) the ground state becomes FM again (Fig. 19). The fact that both iron and niobium-rich samples are FM at low temperature has been linked to the peculiar electronic structure of this material (Subedi and Singh, 2010; Tompsett et al., 2010; Alam and Johnson, 2011; Neal, Ylvisaker, and Pickett, 2011). Part of the phase diagram can also be reproduced by applying hydrostatic pressure (Duncan et al., 2010): Starting with a FM sample with \(y = 0.015\), increasing pressure is equivalent to moving the system to the left in the phase diagram of Fig. 19. A pressure of 2.5 GPa roughly corresponds to a shift in composition from \(y = 0.015\) to 0.007. For \(y \geq -0.015\), Nb\(_{1-y}\)Fe\(_{2+y}\) is thus an itinerant system in which a SDW state with a small wave vector connects continuously to the FM state at a Lifshitz point; cf. Fig. 19. Signatures of quantum-critical behavior have been observed near the QPT where the SDW order disappears: the electrical resistivity displays a \(T^{3/2}\) power-law behavior, and the specific-heat coefficient \(\gamma\) shows a logarithmic temperature dependence down to 0.1 K (Brando et al., 2008). The latter is consistent with Hertz-Millis-Moriya theory for a clean FM,\(^\text{16}\) but the former disagrees with the \(T^{5/3}\) behavior expected for this case; see Eqs. (3.62) and (3.64). A \(T^{3/2}\) behavior of the resistivity has also been observed in other systems, e.g., in MnSi and ZrZn\(_2\), even far from the QPT, which suggests a more general phenomenon that remains incompletely understood; see Sec. IV.A. In the current case, the proximity to both FM and SDW order may result in fluctuations with different wave vectors that have different effects on the transport and thermodynamic properties, respectively. This might explain the apparent inconsistency between the behavior of the specific heat and the resistivity (see footnote\(^\text{16}\)).

The existence of a QCP indicates that the PM-to-SDW transition is second order, at least at low temperature. There are indications that the FM-to-SDW transition at \(y = 0.015\) is first order (Friedemann, 2015). On the Nb-rich side of the phase diagram, the transition was found to be second order for all samples investigated by Moroni-Klementowicz et al. (2009).

b. An induced-moment magnet: PrPtAl

Another example of a modulated magnetic state in a narrow temperature range above the FM phase is given by PrPtAl. Since the orthorhombic TiNiSi-type structure of PrPtAl is inversion symmetric, the modulated order cannot be due to a Dzyaloshinskii-Moriya interaction. Neutron scattering data in conjunction with a theoretical analysis have been interpreted as indicating that it is a result of quantum-critical fluctuations, in accord with the mechanism reviewed in Sec. IIE (Abdul-Jabbar et al., 2015).

Initial experiments on polycrystals identified a second-order phase transition into a FM state at \(T_C = 5.8\) K with an ordered saturation moment of 1\(\mu_B\)/Pr (Hulliger, 1993; Kitazawa et al., 1998). As in other rare-earth systems, the origin of magnetism in PrPtAl is subtle. Although the ground state of the Pr\(^{3+}\) ion is a nonmagnetic singlet, magnetic moments are induced by an interplay between the exchange interaction and the CEF. This causes strong short-range correlations, which are responsible for the small entropy release (only about 15% of \(R \ln 2\)) below \(T_C\) and significant magnetic anisotropy (Kitazawa et al., 1998).

Neutron scattering experiments on single crystals have revealed two SDW phases just above \(T_C\): Below \(T_1 = 5.85\) K Abdul-Jabbar et al. (2015) found a doubly modulated incommensurate SDW (SDW1), followed by a single incommensurate modulation (SDW2) at a different ordering vector below \(T_2 = 5.5\) K, and eventually the transition into the FM state at \(T_C = 4.7\) K. Both SDW phases are suppressed by a weak magnetic field; this may be the reason why these phases were not seen before. The magnetic structure in the SDW2 phase is an elliptical spiral. Spiral order preceding a FM transition is not uncommon in rare-earth magnets such as, for instance, Tb and Dy, and is usually attributed to a complex interplay between the anisotropy energy and the exchange interaction (Miwa and Yosida, 1961). However, in PrPtAl Abdul-Jabbar et al. (2015) pointed at the lack of apparent nesting vectors which would favor spiral order, the temperature dependence of the ordering wave vector in the spiral phase SDW2, the low critical magnetic fields required to tune between the SDW2 phase and FM, and the second-order nature of the transition at \(T_1\), which contrasts with the first-order nature at \(T_2\) and \(T_C\). They argued that the mechanism behind the spiral formation in SDW2 must involve the strong magnetic fluctuations in the competing ordered states. Starting from the model proposed by Karahasovic, Krüger, and

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\(^\text{16}\)As explained in Sec. III.C.2, in the FM case this does not represent true critical behavior, but it may be observable in a sizable preasymptotic region.
Green (2012) and adding local moments, the strong anisotropy, and weak disorder, they derived a theory that can describe key experimental results observed by neutron scattering and measurements of the magnetoresistivity and the specific heat. Although the full phase diagram has not been accessed experimentally, the proposed phase diagram is similar to that in Fig. 39 without the nematic phase. This material may thus represent a case in which the mechanism discussed in Sec. III.E is realized.

2. Ferromagnetic Kondo-lattice systems: CeTPO

Quantum criticality in FM Kondo-lattice systems has received little attention compared to their AFM counterparts (Gegenwart, Si, and Steglich, 2008 and references therein.) Often these materials possess peculiar crystal structures, such as the quasi-1D heavy-fermion material YbNi₄P₂ (Sec. II.C.3), or the quasi-2D cerium transition-metal (T) phosphide oxides CePTO, which are the topic of this section. For other Kondo-lattice systems, see Sec. II.D.3.

The quasi-2D tetragonal ZrCuSiAs-type of the CePTO systems is familiar from some of the iron-based superconductors, such as LaFePO (Kamihara et al., 2006). It consists of alternating layers of TP₄ and OCe₄ along the crystallographic c axis (Zimmer et al., 1995). The Ce-Ce interatomic distance is in the range where the RKKY interaction is ferromagnetic (Sereni, 1991; Chevalier and Malaman, 2004). However, not all CePTO systems are FM, for instance, CeOsPO is an AFM (Krellner et al., 2007). Two compounds that have been studied with respect to FM quantum criticality are CeRuPO and CeFePO. CeRuPO is a ferromagnet with \( T_C = 15 \text{ K} \) (Krellner et al., 2007), and CeFePO is a paramagnet with strong in-plane FM fluctuations (Brüning et al., 2008). In what follows we review studies of CeRuPO under hydrostatic pressure, and of CeRu₁₋ₓFeₓPO and CeFeAs₁₋ₓPₓO. The special case of stoichiometric CeFePO is discussed in Sec. II.E.4.

a. CeRuPO

The low-T properties of CeRuPO polycrystalline samples (RRR = 50) were first investigated by Krellner et al. (2007). They found a Curie-Weiss behavior of the susceptibility at high \( T \) (Ce is trivalent in this compound and Fe is nonmagnetic) with a positive Weiss temperature \( \Theta_W = 8 \) K and FM order below \( T_C = 15 \) K. The transition at \( T_C \) is second order, indicated by the \( \lambda \)-like shape of the specific heat. The resistivity shows a distinctive drop below about 50 K which is a signature of coherent Kondo scattering. The Kondo temperature \( T_K \approx 10 \) K, estimated from an analysis of the entropy, is comparable with \( T_C \).

Using a Sn-flux method, Krellner et al. (2007) grew high-quality single crystals with \( \rho_0 = 5 \mu \Omega \text{cm} \) (RRR = 30) (Krellner and Geibel, 2008), and studied the magnetic anisotropy. At high \( T \) the susceptibility measured with \( H \parallel c \) and \( H \perp c \) shows a Curie-Weiss behavior, but with very different Weiss temperatures for the two cases; \( \Theta_W^c \approx 4 \) K and \( \Theta_W^\perp \approx -250 \) K. This temperature difference can be expressed in terms of the first CEF parameter, which is a measure of the strength of the magnetocrystalline anisotropy (Bowden, Bunbury, and McCausland, 1971). It indicates that the CEF anisotropy favors the moments to be aligned within the \( ab \) plane. However, the magnetic moments below \( T_C \) align along the \( c \) axis. This is shown in Fig. 20, which displays the magnetization isotherms at 2 K. Ordering of the moments along the magnetic hard direction is also found in YbNiSn (Bonville et al., 1992), Yb(Rh₁₋ₓCoₙₓ)₂Si₂ (Lausberg et al., 2013), YbNi₄P₂ (Steppke et al., 2013), and CeAgSb₂ (Araki et al., 2003).

Quantum criticality in CeRuPO has been looked for by means of resistivity measurements under pressure, with the current in the \( ab \) plane. Pressure was found to decrease \( T_C \) to 5.9 K at 2.1 GPa, which by extrapolation suggested a QCP at about 3.2 GPa (Macovei et al., 2009). Lengyel et al. (2015) investigated the ac susceptibility under pressure and performed resistivity experiments at pressures up to 7.5 GPa. These experiments found that the FM ground state changed into an AFM one (with unknown structure) at a pressure of about 0.87 GPa. At \( p \geq 3 \) GPa the resistivity no longer shows a phase transition, and the observations suggested a first-order QPT at a critical pressure \( p_c \approx 3 \) GPa, with no QCP. Above \( p_c \), the ground state was proposed to be a Fermi liquid, due to a \( T^2 \) behavior of the resistivity. The coefficient of the \( T^2 \) term in the resistivity shows a maximum at about 4 GPa, well inside the FL paramagnetic state.

The \( T-p-H \) phase diagram of CeRuPO, up to \( p = 3.5 \) GPa, was investigated by Kotegawa et al. (2013); see Fig. 21. The observed sensitivity of the magnetic order to a magnetic field ascertained that the FM state changes into an AFM one above \( p \approx 0.7 \) GPa, and the magnetic order was found to be completely suppressed at \( p_c = 2.8 \) GPa. CeRuPO is thus another case where an anticipated FM QPT is not realized because a modulated phase intervenes. Although the authors could not determine the order of the transition at \( p_c \), the coefficient of the \( T^2 \) term in the resistivity shows a maximum around \( p_c \), which suggests the presence of an AFM QCP. This is in disagreement with Lengyel et al. (2015). ⁳¹P-NMR experiments revealed the fact that the magnetic correlations are 3D over the entire pressure range investigated (Kitagawa et al., 2014), in contrast to what was found in Ce(Rh₁₋ₓFeₓ)PO; see Sec. II.D.2.c. The FM-AFM transition was found to be first order at the two points denoted by open circles in Fig. 21; the order of the transition at lower
temperatures is not known. We note that the AFM-to-FM transition in Yb(Rh$_{1-x}$Co$_x$)$_2$Si$_2$ with $x = 0.215$ (see Fig. 27) has also been reported to be first order (Klingner et al., 2011; Hamann, 2015), as has the SDW-to-FM transition in Nb$_{1-y}$Fe$_{2+y}$ (see Fig. 19) (Friedemann, 2015).

At pressures close to $p_c$, Kotegawa et al. (2013) observed another resistivity feature at a magnetic field $H_m$ and a temperature $T_0$, which increases with increasing field (see Fig. 21). They ascribed this to a metamagnetic crossover from a PM state to a polarized PM (PPM) state. This is similar to what was observed in doped CeRu$_2$Si$_2$ (Flouquet et al., 2002, 2010; Shimizu et al., 2012).

b. CeFeAs$_{1-x}$P$_x$O

CeFePO is a PM very close to a FM instability (Brüning et al., 2008), which motivated searches for an FM phase nearby. The substitution of As at the P site acts as negative chemical pressure and favors magnetic order in Ce-based systems. It also permits one to study the evolution of Fe and chemical pressure and favors magnetic order in Ce-based nearby. The substitution of As at the P site acts as negative chemical pressure and favors magnetic order in Ce-based systems.

Luo et al. (2010) measured the phase diagram of polycrystalline samples of CeFeAs$_{1-x}$P$_x$O and confirmed that the commensurate AFM order of the Fe sublattice below $T_{AFM} \approx 140$ K in CeFeAsO is suppressed by P substitution. With increasing $x$ the unit cell shrinks along the $c$ axis substantially faster than along the $a$ axis, which is important for the evolution of the $f$-$d$ hybridization strength. The AFM order disappears at $x \approx 0.4$, and an AFM QCP was suspected (Luo et al., 2010; de la Cruz et al., 2010). Moreover, Luo et al. (2010) found that the Ce sublattice also orders antiferromagnetically at $T_{AFM} = 4.16$ K, which is very weakly $x$ dependent for small $x$. At $x \approx 0.37$ the ground state of the Ce sublattice changes from AFM to FM, and FM order was found to persist up to $x \approx 0.9$. In a small concentration region at $x \lesssim 1$ the system was found to be a HF paramagnet with strong FM fluctuations, in agreement with Brüning et al. (2008), and a second QCP, for the FM-HF transition, was suggested.

Later studies by Jesche (2011, 2012) found additional features in the phase diagram (see Fig. 22) and disagree in some respects with Luo et al. (2010). Instead of an AFM QCP they found that the AFM order terminates at a nonzero $T_{AFM} \approx 30$ K and is followed by a region of phase separation, indicating a possible tricritical point. They also found superconductivity, possibly coexisting with Ce ferromagnetism, in a small dome around $x = 0.3$ at temperatures up to 4 K. On the large-$x$ side of the phase diagram Jesche (2011) and Krellner and Jesche (2014) found that a single-crystal sample with $x = 0.9$ had an AFM ground state rather than a FM one. This evidence is shown in the inset of Fig. 22, which shows the isothermal magnetization at $T = 2$ K, i.e., below the transition temperature of about 2.7 K for this concentration. There is no remnant magnetization at $B = 0$ in both field directions, and a metamagnetic transition at $B = 1$ T implies that the ground state is indeed AFM. This indicates that CeFeAs$_{1-x}$P$_x$O belongs to the class of systems where the order changes from FM to AFM as $T_C$ decreases, as is the case in CeRuPO under pressure (Sec. II.D.2.a), and CeRu$_{1-x}$Fe$_x$PO (Sec. II.D.2.c). This implies that there must be a Lifshitz point on the phase boundary shown in Fig. 22, and a QPT from the FM phase to...
the AFM phase, followed by an AFM QPT. These issues have not been investigated.

c. CeRu\textsubscript{1−x}Fe\textsubscript{x}PO

CeRuPO is a low-temperature FM (see Sec. II.D.2.a), while CeFePO is a PM with strong FM fluctuations (see Sec. II.D.2.b); this motivated the study of the series CeRu\textsubscript{1−x}Fe\textsubscript{x}PO. The substitution of Fe for Ru is isoelectronic and affects just the Fe(Ru)P layers without causing much disorder in the CeO layers responsible for the magnetism. Kitagawa et al. (2012) investigated polycrystalline (oriented powder) samples by \textsuperscript{31}P NMR. They measured the Knight shift \(K\), which is proportional to the uniform magnetization, and the spin-lattice relaxation rate \((1/T_1)\), which is a measure of the fluctuations perpendicular to the applied field direction, for fields parallel (\(\parallel\)) and perpendicular (\(\perp\)) to the \(c\) axis of the tetragonal crystallographic structure. Because of the \(XY\)-type anisotropy, the largest signal is found for \(K_\perp\) and \((1/T_1)_\parallel\), which both show a strong \(T\) dependence.

The FM transition temperature was determined by the large increase of \(K(T)\) at \(T_C\) and the peak in \((1/T_1)_\parallel\) as shown in Fig. 23 for \(\mu_0 H = 0.5\) T; the resulting phase diagram is shown in Fig. 24. With increasing Fe content, both \(T_C\) and the ordered moment, as determined from the Knight shift (not shown), are continuously suppressed until both vanish at \(x_c \approx 0.86\). The phase transition is clearly second order at \(x = 0\) and remains second order for all concentrations, although a significant broadening of the relaxation rate and of the Knight-shift increase is seen at \(x = 0.85\). This might indicate a spin-glass-like or short-range ordered state, as was observed in pure CeFePO (Lausberg et al., 2012), see Sec. II.E.4, but this is not quite clear. It is possible that short-range order, if it is present, is suppressed by the applied field (about 0.5 T is needed for most NMR measurements); in CeFePO the short-range ordered state is suppressed at this field strength. One thus must keep in mind that Fig. 24 does not show a zero-field phase diagram, and this is more relevant for \(x \approx x_c\) than for small \(x\).

The strong fluctuations observed near \(x_c\) are a clear sign of a QCP, the nature of which is not purely ferromagnetic. The NMR data in a field \(\mu_0 H = 0.07\) T for a sample with \(x = 0.85\) showed that, in addition to the homogeneous FM \((q = 0)\) component of the fluctuations that levels off below \(T = 3\) K, there are AFM \((q \neq 0)\) components that continue to increase as the temperature is lowered toward \(T_C \approx 300\) mK (Kitagawa et al., 2012). Similar behavior was found in YbRh\textsubscript{2}Si\textsubscript{2} near an AFM QCP (Ishida et al., 2003). The behavior near the QCP was further investigated by Kitagawa et al. (2013), who concluded that the suppression of ferromagnetism is due to a change of the effective dimensionality of the FM fluctuations from 3D to 2D near \(x_c\). This is in contrast to what was observed in stoichiometric CeRuPO under hydrostatic pressure, where the magnetic correlations remain 3D at all values of the pressure, including \(p_c\) (Kitagawa et al., 2014).

Another interesting effect is a metamagnetic crossover (not a first-order transition) at a field \(H_M\) perpendicular to the \(c\) axis (Kitagawa et al., 2011, 2012). The tips of these “crossover wings” coincide with the QCP and the wings grow with increasing field; their shape is thus very different from that of the tricritical wings discussed in Sec. II.B. A possibly related observation is that pure CeFePO at \(\mu_0 H \approx 4\) T shows NFL behavior commonly associated with a QCP (Kitagawa et al., 2011). They suggested that \(H_M\) represents the field that breaks the local Kondo singlet,

![FIG. 23. Temperature dependence of the in-plane spin fluctuations \(S_\parallel = (1/2T_1)|_{\mu_0 H = 0.5\ T}\) for various values of \(x\). The peak indicates the FM transition temperature \(T_C\), which is visible only for \(x \leq 0.85\). For \(x > 0.85\), \(S_\parallel\) is constant at low \(T\). The inset shows the \(x\) dependence of \(S_\parallel\) at a temperature of 200 mK. \(S_\parallel\) peaks at \(x_c\), indicating the presence of a QCP. From Kitagawa et al., 2012.](image)

![FIG. 24. Lower panel: \(x-T\) phase diagram of CeRu\textsubscript{1−x}Fe\textsubscript{x}PO derived from NMR studies. Strong fluctuations are indicative of a QCP at \(x_c \approx 0.86\). At larger \(x\) the ground state is a heavy-fermion (HF) paramagnet, bounded by the temperature \(T_{\text{max}}\) where the Knight shift shows a peak. Upper panel: The exponent \(n\) is indicative of the effective dimensionality of magnetic correlations, with \(n = 1.5\) corresponding to 2D correlations. From Kitagawa et al., 2013.](image)
and that the critical behavior is driven by the Kondo breakdown accompanied by a Fermi-surface instability.

These NMR results paint a picture that is rather different from that of CeRuPO under pressure, where a phase of an AFM character intervenes before the FM QCP is reached; see Sec. II.D.2.a. However, Jesche et al., 2016 found an AFM ground state between the FM and PM phases in single crystals with a Ru content close to 20%, just as in CeRuPO under pressure. An example is shown in Fig. 25 that displays the field dependence of the magnetization for a sample with 22% Ru content at \( T \approx 1.8 \) K, below the transition temperature of 2.5 K for this concentration. There is no remanent magnetization at zero field in either field direction. In addition, for \( H \parallel c \) a metamagnetic increase of the magnetization is found for \( \mu_0 H \approx 1.1 \) T (see the inset in Fig. 25) indicating a first-order metamagnetic transition. This is reminiscent of the situation in CeFeAs\(_1_{-x}\)P\(_x\)O (cf. the inset in Fig. 22) and NbFe\(_2\) (Moroni-Klementowicz et al., 2009).

3. Other Kondo-lattice systems

a. CeAgSb\(_2\)

Neutron scattering experiments on the Kondo-lattice system CeAgSb\(_2\) found an ordered moment of \( 0.4\mu_B/\text{Ce} \) that aligns uniaxially along the tetragonal \( c \) axis, whereas magnetization measurements indicate a strong magnetocrystalline anisotropy with the basal plane as the magnetic easy plane (Araki et al., 2003; Takeuchi et al., 2003). For other examples of such alignment along the hard direction see Sec. II.D.2.a. For this reason, quantum criticality in CeAgSb\(_2\) has been studied by transversal-field tuning with \( H \perp c \). The critical field is \( H_c \approx 2.8 \) T (Strydom, Mhlungu, and Thamizhavel, 2008; Logg et al., 2013; Zou et al., 2013). The transition is suspected to remain second order to the lowest \( T_C \) measured, about 2 K.

b. CeRu\(_2\)Ge\(_2\) and CeRu\(_2\)(Ge\(_{1-x}\),Si\(_x\))\(_2\)

CeRu\(_2\)Ge\(_2\) is a FM Kondo-lattice system with \( T_C \approx 8 \) K, a spontaneous magnetization of \( 1.96\mu_B \) along the tetragonal \( c \) axis (Bohm et al., 1988; Besnus et al., 1991), a Kondo temperature \( T_K \approx 2 \) K, and a rather small Sommerfeld coefficient of 20 mJ/K\(^2\) mol. In some crystalline samples specific-heat and magnetization measurements exhibit two transitions, a hump at \( T_N \approx 8.2 \) K and a sharp transition at \( T_C \approx 7.7 \) K (Fontes et al., 1996; Raymond, Raehelson et al., 1999). Between \( T_C \) and \( T_N \), long-range order was identified to be AFM with an incommensurate wave vector \( q = (0.31, 0, 0) \).

Studies under hydrostatic pressure (Kobayashi et al., 1998; Wilhelm and Jaccard, 1998; Sullow et al., 1999) revealed a rich \( p-T \) phase diagram that is well reproduced by Si substitution for Ge (Haen, Bioud, and Fukuhara, 1999). Both local-moment FM and AFM phases exist at low pressures. Above 20 kbar the FM phase changes into a second low-\( T \) AFM phase. The magnetic order is then rapidly suppressed near \( p_c \approx 67 \) kbar, accompanied by NFL behavior.

FIG. 25. Isothermal magnetization of a single crystal of CeRu\(_{0.22}\)Fe\(_{0.78}\)PO for \( H \parallel c \) and \( H \perp c \) at \( T \approx 1.8 \) K. Note the absence of a remanent magnetization at \( H = 0 \). The inset shows a weak hysteresis loop near \( \mu_0 H = 1.1 \) T for \( H \parallel c \). From Jesche et al., 2016.

FIG. 26. Pressure-temperature phase diagram of CeAgSb\(_2\). \( T_C \) and \( T_N \) were determined from \( dp/dT \) while \( T_{mag} \) is from ac-calorimetry measurements. From Sidorov et al., 2003.
with a linear-in-$T$ resistivity. For $p > p_c$ the ground state is a Fermi liquid with an enhanced quasiparticle mass that decreases toward higher pressures (Süllow et al., 1999). A similar phase diagram was obtained by substituting Ru by Fe (Raymond, Raoelison et al., 1999).

Although the $p$-$T$ phase diagram of CeRu$_2$Ge$_2$ and its Si-doped variety has the general shape shown in Fig. 2(c), the underlying physics may be different from other systems. This is because the change in the ground state from FM to AFM occurs when the Kondo temperature is much smaller than the transition temperatures (Süllow et al., 1999). The modification of the ordered state is therefore likely due to a pressure dependence of the exchange interaction and is not driven by the mechanism discussed in Sec. III.E.

c. YbRh$_2$Si$_2$ and Yb(Rh$_{1-x}$Co$_x$)$_2$Si$_2$

The heavy-fermion metal YbRh$_2$Si$_2$ (Trovarelli et al., 2000) is a prototypical example of a quantum-critical system (Gegenwart et al., 2002; Custers et al., 2003); for reviews, see Gegenwart, Si, and Steglich (2008) and Si and Steglich (2010). Here we focus only on properties that are related to ferromagnetism.

YbRh$_2$Si$_2$ crystallizes in the body-centered tetragonal ThCr$_2$Si$_2$ structure. The Yb ions are in the trivalent state as indicated by the high-$T$ Curie-Weiss behavior of the susceptibility $\chi(T)$ with an effective magnetic moment of $4.4\mu_B$, i.e., close to what is expected for a free Yb$^{3+}$ ion. The Weiss temperatures $\Theta_{\chi}(B/c) = -180 \, \text{K}$ and $\Theta_{\chi}(B/Lc) = -9 \, \text{K}$ indicate a strong magnetocrystalline anisotropy (Trovarelli et al., 2000). The CEF splits the $J = 7/2$ levels into four Kramers doublets, leaving the ground state separated from the three excited doublets by approximately 17, 25, and 43 meV, respectively (Stockert et al., 2006). YbRh$_2$Si$_2$ has a high Kondo temperature $T_K \approx 25 \, \text{K}$ (Köhler et al., 2008), but a small unscreened magnetic moment of about $0.2\mu_B$/Yb (Ishida et al., 2003) orders antiferromagnetically below $T_N \approx 0.07 \, \text{K}$ (Trovarelli et al., 2000). The exact magnetic structure is still unknown. $T_N$ can be suppressed by a magnetic field $B \approx 0.06 \, \text{T}$ perpendicular to the magnetically hard $c$ axis (Gegenwart et al., 2002), or by negative chemical pressure ($p \approx -0.25 \, \text{GPa}$) (Mederle et al., 2001; Macovei et al., 2008), which tunes the system to QCPs.

YbRh$_2$Si$_2$ shows pronounced NFL behavior in transport and thermodynamic quantities, indicating the presence of strong spin fluctuations. For instance, the resistivity $\rho(T) \propto T$ below 10 K and the Sommerfeld coefficient diverges as a power law $C/T \propto T^{-0.34}$ for $T \lesssim 0.3 \, \text{K}$, similar to what is observed in the low-$T$ FM YbNi$_3$P$_2$ (Krellner et al., 2009). The latter behavior has been interpreted in terms of a breakup of the heavy quasiparticles at the QCP (Custers et al., 2003). At low temperature the susceptibility $\chi_{\perp c}$ is very large for an AFM ($\approx 8.5 \times 10^{-6} \, \text{m}^3/\text{mol} \approx 0.18 \, \text{SI}$) and about 20 times larger than $\chi_{\parallel c}$. This and the value of the Sommerfeld-Wilson ratio of about 30 indicate the presence of strong FM fluctuations (Gegenwart et al., 2005), consistent with NMR (Ishida et al., 2003) and neutron scattering data (Stockert et al., 2012). In an intermediate temperature range, for $0.3 \lesssim T \lesssim 4 \, \text{K}$, $C(T)/T \propto \ln(T_0/T)$, with $T_0 \approx 25 \, \text{K}$ the characteristic spin-fluctuation temperature according to Moriya (1985). Pressure stabilizes the magnetic order, increasing both $T_N$ and the ordered moment (Mederle et al., 2001; Knebel et al., 2006). At small pressures an additional transition is observed at a lower temperature $T_L$, which moves toward $T_N$ with increasing pressure. At about 5 GPa, $T_L \approx T_N$ and Knebel et al. (2006) proposed a FM ground state above 5 GPa in YbRh$_2$Si$_2$.

Isoelectronic substitution of Rh by Co in Yb(Rh$_{1-x}$Co$_x$)$_2$Si$_2$ leads to a similar effect as pressure; the correspondence is excellent for $p \leq 2.5 \, \text{GPa}$ (Klingner et al., 2011). The phase diagram for $x < 0.3$ is shown in Fig. 27. At $x = 0.27$, which corresponds to about 4.5 GPa, $T_L = T_N = 1.3 \, \text{K}$. In this sample FM order was indeed found by Lausberg et al. (2013). The moments order along the magnetically hard $c$ axis, similar to YbNi$_3$P$_2$ (Steppke et al., 2013) and CeRuPO$_3$ (Krellner et al., 2007), and despite the large magnetocrystalline anisotropy (which is about 6 in Yb(Rh$_{0.73}$Co$_{0.27}$)$_2$Si$_2$). A sharp peak at $T_C = 1.30 \, \text{K}$ and $B = 0$ is suppressed and shifted toward higher $T$ with increasing field. (c) Temperature dependence of the imaginary part $\chi''(T)$ of the susceptibility. From Lausberg et al., 2013.

FIG. 27. (a) $T$-$x$ phase diagram of Yb(Rh$_{1-x}$Co$_x$)$_2$Si$_2$ as measured by Kplingner et al. (2011) showing the transition temperatures $T_L$ and $T_N$. The red arrow marks the sample with $x = 0.27$ which is FM. (b) Real part $\chi'(T)$ of the susceptibility for Yb(Rh$_{0.73}$Co$_{0.27}$)$_2$Si$_2$ in different magnetic fields with $B/c$. The sharp peak at $T_C = 1.30 \, \text{K}$ and $B = 0$ is suppressed and shifted toward higher $T$ with increasing field. (c) Temperature dependence of the imaginary part $\chi''(T)$ of the susceptibility.
As in CeRu$_2$Si$_2$, the physics of Yb(Rh$_{1-x}$Co$_x$)$_2$Si$_2$ is controlled by the evolution of the Kondo temperature, the magnetic anisotropy, and the exchange interactions, and not simply by the mechanism discussed in Sec. III.E that could be valid for a simpler system such as Nb$_{1-y}$Fe$_{2+y}$.

4. Discussion, and comparison with theory

All of the phase diagrams discussed in this section have the same overall structure: As the Curie temperature decreases, the ground state changes from a homogeneous FM to some modulated magnetic state that is often summarized referred to as AFM, even if its detailed structure is not known. This is an important distinction from a theoretical point of view, as classic AFM involves structure on an atomic scale, whereas other modulated states, such as the SDW in Nb$_{1-y}$Fe$_{2+y}$, Sec. II.D.1.a, or the helimagnetism in MnSi, Sec. II.B.1.a, are long-wavelength phenomena. The general structure of these phase diagrams is shown schematically in Fig. 2(c). It contains two QPTs (between the FM and modulated states, and between the modulated and nonmagnetic states, respectively), and a multicritical point where the three phases meet.

The issues related to the nature and properties of AFM QCPs are very different from those of FM QCPs. Most of the theoretical concepts discussed in Sec. III, with the exception of Sec. III.E, apply to FM QCPs only, and questions related to the first-order versus second-order nature of any AFM QPT must not be confused with the corresponding questions for FM QPTs. Moreover, in some systems, especially Kondo-lattice systems, such a change from FM to AFM order can occur as the result of competing interactions, i.e., frustration, as a result of pressure or chemical substitution. These mechanisms are very different from the one discussed in Sec. III.E.

An interesting topic is the order of the QPT from the FM to the modulated phase. FM-to-AFM transitions are common even at high temperatures and are often accompanied by a structural phase transition, which makes them first order. The nature of the FM-to-AFM and SDW transitions discussed in this section is probably different, as indicated by the small wave number that characterizes the modulated phase. They likely belong to the class of Lifshitz transitions, which separate a homogeneous phase from a phase with a modulated order parameter [see Hornreich, Luban, and Shtrikman (1975) and references therein, and Chaikin and Lubensky (1995)], and have been considered in many contexts. On the basis of the mechanism discussed in Sec. III there are theoretical reasons to believe that a QPT from a metallic FM to a modulated magnet is generically of first order, although a detailed theory remains to be worked out; see Sec. IV.B. Classically, this transition can be either first or second order (Chaikin and Lubensky, 1995). A generalization of the theory reviewed in Sec. III.B that allows for a modulated order parameter has been developed by Karahasano, Krüger, and Green (2012) [see Fig. 39 (the nematic phase may or may not be present) and the discussion in Sec. III.E]. The structure of the resulting phase diagram agrees with what is observed in, e.g., Nb$_{1-y}$Fe$_{2+y}$; see Fig. 19. However, the theory predicts a first-order transition from the AFM or spiral phase to the PM, whereas the SDW-PM transition in Nb$_{1-y}$Fe$_{2+y}$ is observed to be continuous.

FM Kondo-lattice systems have been studied theoretically by Perkins et al. (2007) by means of a mean-field theory, and by Yamamoto and Si (2010) by means of a RG treatment. Both found a second-order QPT, i.e., the RG treatment found that the earlier generalized Stoner theory is exact with respect to the order of the transition. Technically, this is because the calculation by Yamamoto and Si (2010) does not yield the nonanalytic wave-number dependence of the spin propagator found in a related model (Chubukov, Pépin, and Rech, 2004), which destroys the FM QCP and leads to a first-order transition as described in Sec. III.B.2. While it is conceivable that the two models are different in this respect, this seems unlikely. There is reason to believe that the theory reviewed in Sec. III, if applied to Kondo-lattice systems, will yield a first-order transition. However, this has not been worked out in detail and more work on this topic is needed. In any case, these theories consider a QPT from a FM metal to a PM metal, which so far has not been observed in Kondo-lattice systems (although in some materials the experimental situation is not quite clear yet, see, e.g., CeRu$_{1-x}$Fe$_x$PO in Sec. II.D.2.c).

E. System showing glasslike behavior, short-range order, or other strong-disorder effects

This section (with the exception of Sec. II.E.5) describes FM metallic systems that display effects believed to be characteristic of strong disorder in the region where the FM order is destroyed, and for many of them the nature and precise location of the FM QPT is not clear. Some materials display effects that have been interpreted as evidence for a quantum Griffiths region on the PM side of the QCP, with or without glassy freezing of the rare regions or clusters that characterize the Griffiths region. A special case is CeFePO, which displays short-range magnetic order in the absence of strong quenched disorder. The systems discussed here are listed in Table VII. They are arranged with respect to their phenomenology and/or its interpretation.

The behavior characteristic of these materials can often be obtained by substituting a magnetic element by a nonmagnetic one of the same series, e.g., uranium by thorium in U$_{1-x}$Th$_x$Ni$_2$S$_2$, or nickel by vanadium in Ni$_{1-x}$V$_x$. This usually introduces substantial amounts of quenched disorder. The x-T phase diagram is often characterized by a pronounced tail [cf. Fig. 2(d)], and the region immediately above the tail is generally characterized by NFL behavior. The tail has been interpreted in terms of locally ordered clusters, and the NFL behavior in terms of quantum Griffiths singularities, a topic reviewed in Sec. III.D. This is, however, not the only possible explanation for a tail in the phase diagram; see the discussion in Sec. III.B.3. Some of the theoretical and experimental results pertinent to this section have been summarized by

\[17\text{ The evidence for an AFM phase is stronger in some materials than in others, and additional experimental work is needed in many cases; see the earlier discussions of the individual systems.} \]

\[18\text{ We do not include diluted magnetic semiconductors, such as Fe}_{1-x}Co}_xS_2. For an example, see Guo et al. (2008). \]
TABLE VII. Systems showing short-range order or spin-glass (SG) freezing. $T_C =$ Curie temperature, $T_g =$ freezing temperature, $\rho_0 =$ residual resistivity, QC = quantum critical, QGP = quantum Griffiths phase, N/A = not applicable, and n.a. = not available.

<table>
<thead>
<tr>
<th>System</th>
<th>Order of transition</th>
<th>$T_C$/K</th>
<th>$T_g$/K</th>
<th>Magnetic moment/(\mu_B)</th>
<th>Tuning parameter</th>
<th>Disorder $\rho_0/\mu\Omega$ cm</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>CePd$_{1-x}$Rh$_x$</td>
<td>n.a.</td>
<td>$6.6 - 3^d$</td>
<td>$3 - 0^b$</td>
<td>n.a.</td>
<td>Composition$^{12}$</td>
<td>n.a.</td>
<td>Kondo cluster glass</td>
</tr>
<tr>
<td>CePt$_{1-x}$Rh$_x$</td>
<td>n.a.</td>
<td>$6 - 2^d$</td>
<td>$3 - 2^b$</td>
<td>n.a.</td>
<td>Composition$^4$</td>
<td>n.a.</td>
<td>Cluster glass (?)$^b$</td>
</tr>
<tr>
<td>Ni$_{1-y}$V$_y$</td>
<td>n.a.</td>
<td>$633 \approx 30^{1/4}$</td>
<td>$0.6 - 0$</td>
<td>n.a.</td>
<td>Composition$^3$</td>
<td>n.a.</td>
<td>cluster glass (?)$^b$</td>
</tr>
<tr>
<td>UNi$_{1-x}$Co$_x$Si$_2$</td>
<td>n.a.</td>
<td>$95 - 8.6^{2/3}$</td>
<td>$6 \chi^{1/2}$</td>
<td>n.a.</td>
<td>Composition$^1$</td>
<td>15 $\mu\Omega$ cm$^m$,&amp;</td>
<td>Glassy phase?</td>
</tr>
<tr>
<td>U$_{1-x}$Th$_x$NiSi$_2$</td>
<td>n.a.</td>
<td>$95 - 29^{a,7}$</td>
<td>$29 - 4^{a,7}$</td>
<td>n.a.</td>
<td>Composition$^7$</td>
<td>n.a.</td>
<td>Possible QGP</td>
</tr>
<tr>
<td>CeTi$_{1-x}$V$_x$Ge$_3$</td>
<td>2m$^9$</td>
<td>$14 - 2.8^{8,9}$</td>
<td>$1.5^9$</td>
<td>Composition$^9$</td>
<td>$22^{9,0}$</td>
<td>QGP?</td>
<td></td>
</tr>
<tr>
<td>Sr$_{1-x}$Ca$_x$RuO$_3$</td>
<td>n.a.</td>
<td>$\approx 100 - 40^{10}$</td>
<td>$40 - 5^{10}$</td>
<td>1 - 0</td>
<td>Composition$^{10}$</td>
<td>$8^9$</td>
<td>Thin films</td>
</tr>
</tbody>
</table>

CeFePO N/A N/A 0.9$^{11}$ n.a. None 800$^{12}$ Low intrinsic disorder

$^a$For the disappearance of homogeneous FM order.
$^b$Per formula unit unless otherwise noted.
$^c$For the highest-quality samples.
$^d$For $x = 1 - 0.6$ (Sereni et al., 2007)
$^e$For $x = 0.6 - 0.9$ (Sereni et al., 2007; Westerkamp et al., 2009).
$^f$For $x = 0 - 0.7$
$^g$For $x = 0.5$ and 0.6.
$^h$See footnote $^i$.
$^i$For $x = 0 - 0.105$ (Ubaid-Kassis, Vojta, and Schroeder, 2010).
$^j$For $x = 0.11 - 0.1225$ (Ubaid-Kassis, Vojta, and Schroeder, 2010).
$^k$For $x = 0 - 0.96$ (Pikul and Kaczorowski, 2012).
$^l$For $x = 0.98$ (Pikul and Kaczorowski, 2012).
$^m$For UNiCoSi$_x$
$^n$For $x = 0 - 0.7$ (Pikul, 2012).
$^o$For $x = 0.7 - 0.9$ (Pikul, 2012).
$^p$For CeTiGe$_3$.
$^q$For $x = 0.15 - 0.38$ (Demko et al., 2012).
$^r$For $x = 0.38 - 0.52$ (Demko et al., 2012).
$^s$For $x = 0$ (Schneider, Mosshyna, and Gegenwart, 2010). RRR values varied from a high of 28.9 for $x = 0$
to a low of 2.9 for $x = 0.5$.
$^t$High $\rho_0$ not intrinsic, but due to granularity of the polycrystalline sample.
$^u$Sereni et al. (2007).
$^v$Westerkamp et al. (2009).
$^w$Kawasaki et al. (2008, 2009).
$^x$Ubaid-Kassis, Vojta, and Schroeder (2010).
$^z$Kaczorowski (1996).
$^{10}$Pikul (2012).
$^{11}$Manfrinetti et al. (2005).
$^{12}$Kittler et al. (2013).
$^{13}$Demko et al. (2012).
$^{14}$Lausberg et al. (2012).
$^{15}$Brüning et al. (2008).

Vojta (2010). As in the case of Sec. II.D, different experiments and their interpretations are not consistent for some materials, and it is possible that some of the systems discussed later will eventually be classified with those in Sec. II.C. Conversely, some materials discussed in Sec. II.C may eventually be found to belong in the current section, especially URu$_2$Re$_2$Si$_2$, Sec. II.C.2.b, and possibly Ni$_4$Pd$_{1-x}$, Sec. II.C.1.a, while CeFePO may belong into Sec. II.B according to some experimental results; see Sec. II.E.4.

1. Systems with glasslike features

Systems with strong-disorder effects include CePd$_{1-x}$Rh$_x$, Ni$_{1-x}$V$_x$, and UNi$_{1-x}$Co$_x$Si$_2$. Their behavior has been interpreted in terms of a Griffiths region in the PM phase, with symptoms of glassy freezing at the lowest temperatures.

a. CePd$_{1-x}$Rh$_x$

This series crystallizes in the orthorhombic CrB structure and evolves from a FM ground state in CePd with $T_C = 6.6$ K to a nonmagnetic intermediate-valence state in CeRh (Kappler et al., 1991; Sereni, Beaurepaire, and Kappler, 1993). The chemical substitution of the Ce-ligand Pd with Rh induces not just a volume effect (positive chemical pressure), but also increases the local hybridization strength of the cerium $4f$ electrons with the conduction electrons, leading to a strong enhancement of the Kondo temperature $T_K$ (Sereni et al., 2007). In addition, the Rh substitution introduces disorder.

Evidence for the FM nature of the ordered state comes from the $T$ dependence of the ac susceptibility $\chi'(T)$, which shows large and sharp maxima for all samples ranging from $x = 0.6$ to 0.87 (Sereni et al., 2007). No maximum was observed down to 20 mK in a sample with $x = 0.9$; this indicates a critical concentration for the loss of ferromagnetism very close to $x_c = 0.87$ (Westerkamp et al., 2009). The phase diagram in Fig. 28 shows the transition temperature deduced from measurements of various observables as a function of $x$. Westerkamp et al. (2009) attributed the continuous decrease of $T_C$ with increasing $x$ on the competition between FM order and growing Kondo screening. The curvature of $T_C$ changes from negative to positive at $x \approx 0.6$, displaying a long tail toward higher Rh contents. In this concentration range, the Kondo temperature $T_K \approx 2|\theta_F|$, with $\theta_F$ the paramagnetic Weiss temperature obtained from fits of the dc susceptibility at high temperatures, strongly increases with $x$. The main
mechanism governing $T_K$ is the hybridization of the Ce 4f electrons with the valence electrons of the surrounding ligands. In Ce-based compounds, Rh ligands are known to lead to much larger $T_K$ than Pd ligands (Koepling, Dunlap, and Crabtree, 1985). Thus, in CePd$_{1-x}$Rh$_x$ the effect of the Rh ligands is much stronger than the effect of the Pd ligands once the Rh content reaches a critical value close to 0.7. The random distribution of Rh and Pd ligands likely creates regions with different local values of $T_K$. An analysis of the entropy and the slope of $\chi'(T)$ at 2 K revealed some fraction of unscreened magnetic moments, even at large $x$, where the average $T_K$ is already above 50 K. The pronounced maxima in $\chi'(T)$ of samples with concentrations $x \geq 0.6$ exhibit a frequency dependence similar to that observed in spin glasses (Westerkamp et al., 2009). The relative temperature shift of about 3% to 10% per decade is considerably larger than that in canonical metallic spin glasses (where typical values are 1% to 2%), but well below the value of about 28% observed in superparamagnets (Mydosh, 1993). This behavior and zero-field-cooled and field-cooled magnetization measurements have been interpreted as evidence for the existence of clusters of magnetic moments in the system below a certain temperature $T_{\text{cluster}}$ (Westerkamp et al., 2009). Since the broad distribution of local Kondo temperatures is thought to be responsible for the cluster formation, Westerkamp et al. (2009) called the low-$T$ state in CePd$_{1-x}$Rh$_x$ a “Kondo-cluster glass.” The observation of reentrant depolarization in recent neutron-depolarization imaging experiments (Schmakat et al., 2015) on the same samples investigated by Westerkamp et al. (2009) seems to confirm the presence of such a state. The general behavior of CePd$_{1-x}$Rh$_x$ is quite different from that observed in many other disordered NFL systems, such as CeNi$_{1-x}$Cu$_x$, where the Ce valence remains nearly trivalent and where a percolative cluster scenario was proposed (Marcano et al., 2007).

Specific-heat measurements have shown the existence of NFL behavior for concentrations $0.85 \leq x \leq 0.9$ (Deppe et al., 2006; Pikul et al., 2006). Samples in this concentration range show a power-law dependence $C(T)/T \sim T^{\lambda - 1}$, with exponents $\lambda = 0.6$ and 0.67 for $x = 0.87$ and 0.9, respectively (see inset a of Fig. 29). Power-law behavior has also been found for the $T$-dependent ac susceptibility; see inset b of Fig. 29. These findings suggest that there are strong fluctuations in an entire range of Rh concentrations, which raises the question whether $T_C$ going to zero near $x = 0.87$ represents a FM QCP or not.

A clear answer was given by Westerkamp et al. (2009) who measured the Grüneisen ratio, defined as $\Gamma \propto \beta / C$, with $\beta$ the volume thermal expansion coefficient and $C$ the specific heat. $\Gamma$ must diverge as a power law as $T$ goes to zero at any QCP (Zhu et al., 2003); this was confirmed experimentally for several Kondo-lattice systems exhibiting an AFM QCP (Küchler et al., 2003). Close to the critical concentration, where the anomaly in $\chi'(T)$ disappears, $\Gamma(T) \propto \ln T$ was found, contrary to the power-law divergence expected at a FM QCP. This shows that there is no FM QCP in CePd$_{1-x}$Rh$_x$ at this concentration. Rather, in the region $0.7 < x < 0.9$ and in the temperature range $T_C \leq T \leq T_{\text{cluster}}$ the observations are consistent with the quantum-Griffiths-phase scenario that

---

19The term “cluster glass” is frequently used in a spin-glass context [see, e.g., Iloh et al. (1994)], and different authors use it for various phenomena and concepts whose underlying physics may be quite different.
observed. QCP, which one expects for quantum Griffiths behavior in the PM phase, and the true interpretation, the observations for $0 \leq x \leq 0.9$ represent quantum Griffiths behavior in the PM phase, and the true QCP, which one expects for $x \approx 0.7$, so far has not been observed.

b. CePd$_{1−x}$Rh$_x$

Kawasaki et al. (2008, 2009) found that CePd$_{1−x}$Rh$_x$ behaves similarly to CePd$_{1−x}$Rh$_x$ and successfully fitted the freezing temperature, identified as the temperature where the ac susceptibility shows a pronounced peak, to a Vogel-Fulcher law.

c. Ni$_{1−x}$V$_x$

A small amount of vanadium (about 12%) suppresses $T_C$ to zero from $T_C \approx 630$ K in pure nickel (Bölling, 1968). Ni$_{1−x}$V$_x$ is attractive for studying quantum Griffiths effects for several reasons: (i) it is simpler than Kondo-lattice ferromagnets and has Heisenberg symmetry, (ii) the high $T_C$ of nickel allows the effects to be observable in a larger temperature range than in other systems, and (iii) a vanadium impurity causes a strong reduction (about 90%) of the magnetic moment of the neighboring Ni atoms, which creates significant disorder. This is in contrast to Ni$_{1−x}$Pd$_x$, where isoelectronic Pd substitution does not introduce much disorder, and a large amount of Pd (about 97.5%) is needed to suppress $T_C$ to zero; see Sec. II.C.1.a.

Ubaid-Kassis, Vojta, and Schroeder (2010) measured the magnetization $M$ and the ac susceptibility $\chi$ of several samples with $0 \leq x \leq 0.15$. The $x$-$T$ phase diagram is shown in Fig. 30 (note the log-linear plot). For $x \leq 0.11$ $T_C$ was estimated from standard Arrott plots $H/M = a + bM^2$. For larger $x$ the determination of $T_C$ was model dependent; for $x \geq 0.11$ and fields $H > 0.5$ T a modified Arrott plot $M^{1/\beta} = M_0^{1/\beta}(T) + c(H/M)^{1/\beta}$ was used. Recent $\mu$SR data confirmed that the long-range FM order is lost for $x$ close to 0.11 (Schroeder et al., 2014). In addition to the Arrott plots, the temperature $T_{\text{max}}$ of the maximum in the ac susceptibility was determined as in the case of CePd$_{1−x}$Rh$_x$. The field and frequency dependence of $T_{\text{max}}$ for $x \approx 0.12$ was found to be consistent with what is expected for a cluster glass (see Sec. III.D.3). The dashed line in Fig. 30 is a linear extrapolation of $\ln T_C$ vs $x$, which represents a shape of the phase diagram similar to that in Fig. 2(d).

In the PM region of the phase diagram with $0.114 \leq x \leq 0.15$ the $T$ dependence of the susceptibility $\chi = M/H - \chi_\text{orb}$ (with a small orbital contribution $\chi_\text{orb} = 6 \times 10^{-5}$ emu/mol) can be fitted to a power law $T^{-\gamma}$ for $10 \leq T \leq 300$ K; see FIG. 31. The low-field susceptibility $\chi = M/H - \chi_\text{orb}$ vs $T$, and the magnetization $M = M - \chi_\text{orb}H$ at 2 K vs $H$ for samples with $0.114 \leq x \leq 0.15$. $\chi_\text{orb}$ is the orbital contribution to $\chi$. The dashed lines indicate power-law behaviors $\chi \propto T^{-\gamma}$ and $M \propto H^\gamma$. From Ubaid-Kassis, Vojta, and Schroeder, 2010.

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and Kaczorowski (2012). Cobalt substitution leaves both the uranium lattice and the orthorhombic crystal structure intact, while reducing the unit cell volume by about 1.2%. Since the $b$ axis stretches with increasing $x$, while the $a$ and $c$ axes shrink, Co substitution is not equivalent to hydrostatic pressure. The main effect of the doping seems to be the modification of the intersite coupling between the U magnetic moments: A Curie-Weiss analysis of the susceptibility shows that the effective moment is almost unaffected by the Co concentration the phase boundary changes its curvature and develops a marked tail [cf. Figs. 2(d) and 33]. For the sample with $x = 0.8$, $C(T)/T$ does not show a maximum, as one would expect in a spin glass (Mydosh, 1993), and it does not level off either as in a Fermi liquid, but keeps increasing with decreasing $T$. This anomalous behavior was also observed in other doped FM systems, for instance, CePd$_{1-x}$Rh$_x$, where the thermodynamics in the “tail” region of the phase diagram are believed to be dominated by quantum Griffiths effects [see Westerkamp et al. (2009) and Sec. II.E.1.a].

b. CeTi$_{1-x}$V$_x$Ge$_3$

CeTiGe$_3$ is a FM Kondo-lattice system with $T_C = 14$ K and a hexagonal perovskite BaNiO$_2$-type structure (Manfrinetti et al., 2005). The system CeTi$_{1-x}$V$_x$Ge$_3$ was studied by Kittler et al. (2013). So far only polycrystalline samples with a relatively large residual resistivity $\rho_0 \approx 22 \mu\Omega$cm (for CeTiGe$_3$) have been investigated. At high temperature the susceptibility was reported to follow a Curie-Weiss behavior with a negative Weiss temperature $\theta_W = -36.5$ K, indicating predominantly AFM interactions. The effective moment is $2.64\mu_B$, close to the value of $2.54\mu_B$ for trivalent free +3 Ce ions. The resistivity increases with decreasing temperature, displaying a maximum at about 35 K which indicates the onset of Kondo coherence. The Ce ground state is a Kramers doublet, but the entropy just above $T_C$ is larger than the $R \ln 2$ expected for a doublet ground state, suggesting that the CEF splitting is small and that the Kondo temperature cannot easily be determined from the entropy. The ordered moment measured by neutron powder diffraction within the FM phase is $1.5\mu_B$/Ce, and the ordering is collinear, with moments pointing along the crystallographic $c$ axis (Kittler et al., 2013). The specific heat below $T_C$ suggests the presence of a spin gap $\Delta/k_B \approx 0.8$ $T_C$ in the magnetic excitation spectrum, which indicates a strong magnetic anisotropy.

Nonisoelectronic vanadium substitution for titanium in CeTi$_{1-x}$V$_x$Ge$_3$ permits one to reduce $T_C$ and completely suppress it at $x_c \approx 0.35$, while Ce retains its +3 valence in all samples (Kittler et al., 2013). CeVGe$_3$ shows AFM order
below 4 K (Bie and Mar, 2009). Magnetization measurements demonstrate that the ordered moment is also reduced with increasing $x$. The phase transition is second order down to about 3 K, as indicated by the mean-field-like feature in the specific heat, and remains second order and ferromagnetic even at higher V concentrations, but it broadens strongly for $x$ close to $x_c$. This is an indication of strong-disorder effects. The logarithmic increase of $C_s/T$ toward low temperatures for the $x = 0.3$ sample indicates the presence of spin fluctuations, which might arise from the presence of a QCP at $x_c$ or, more probably, from quantum Griffiths effects, similarly to what was observed in CePd$_{1-x}$Rh$_x$ and Ni$_{1-x}$V$_x$. The experimental phase diagram of CeTi$_{1-x}$V$_x$Ge$_2$ is shown in Fig. 34. The dotted line denotes a linear decrease of $T_C$ with $x$, and the solid line is a fit to the behavior expected from Hertz-Millis-Moriya theory, $T_C \sim (x_c - x)^{3/4}$; see Sec. III.C.2. The lowest temperature achieved was 2 K. Low-$T$ data in the region at $x \approx x_c$ are required to determine whether or not a Griffiths region as discussed in Sec. III.D is indeed present in this system.

3. A thin-film system: Sr$_{1-x}$Ca$_x$RuO$_3$ (thin-film samples)

Bulk (ceramic and powder) samples of Sr$_{1-x}$Ca$_x$RuO$_3$ were discussed in Secs. II.B.4 and II.C.1, respectively. Thin films have been grown epitaxially by Schneider, Moshnyaga, and Gegenwart (2010), Wissinger et al. (2011), and Demko et al. (2012). Schneider et al. found that $T_C$ decreases roughly linearly with $x$, with an extrapolated critical value $x_c \approx 0.7$, in agreement with results on powder, polycrystalline, and ceramic samples. However, Wissinger et al. (2011) found significant differences between film and bulk samples, including a higher value of $x_c$ for films. Demko et al. (2012) measured the magnetization and susceptibility using a magneto-optical technique on a composition-spread epitaxial film of 200 nm thickness. They found a phase diagram that differs markedly from previous results, including those by Schneider, Moshnyaga, and Gegenwart (2010), namely, a pronounced tail with an onset around $x = 0.4$; see Fig. 35. They interpreted their results as the FM-to-PM quantum phase transition being destroyed by the disorder. This is consistent

with a microscopic model which considers spatial disorder correlations at smeared phase transitions (Vojta, 2003) with a behavior different from that at critical points (Demko et al., 2012; Svoboda et al., 2012).

4. A system showing short-range order: CeFePO

CeFePO, a homologue of the quaternary pnictides, is a stoichiometric Kondo-lattice system that is very close to a FM instability (Brüning et al., 2008). However, its ground state is neither FM nor PM, but a short-range ordered state (Lausberg et al., 2012), which is very unusual for a clean system. The first comprehensive low-$T$ study of CeFePO was performed by Brüning et al. (2008), who investigated polycrystals by measurements of the uniform susceptibility, resistivity, NMR (oriented powder) down to 2 K, and specific heat down to 0.4 K. They found that CeFePO is a heavy-fermion system (iron is not magnetic in this compound) with a Kondo temperature $T_K \approx 10$ K, a Sommerfeld coefficient $\gamma = 0.7 J/K^2$ mol, which corresponds to a mass enhancement of 50, a Sommerfeld-Wilson ratio of 5.5, and a Korringa ratio $S_0/T_1TK^2 \approx 0.065$, indicating the presence of FM correlations. Below 10 K the broadening of the line shape of the NMR spectra for small fields $H \perp c$, but not for $H \parallel c$, suggests that in this temperature regime short-range FM correlations that cannot be ascribed to disorder start to be relevant. Thus, only the basal-plane component of the cerium 4f moment is FM correlated. This strong anisotropy reflects the quasi-2D crystal structure. Later, $\mu$SR experiments were performed on polycrystals and single crystals, together with ac susceptibility and specific-heat measurements down to 0.02 K (Lausberg et al., 2012). The ac susceptibility shows a frequency-dependent peak at $T_q \approx 0.9$ K, whose dependence on the modulation frequency is larger than that found in canonical spin glasses and smaller than that of superparamagnets. The entropy measured below the freezing maximum is just 1% of...
longitudinal field is shown in Fig. 36. In conclusion, there is no predictions of textured states in itinerant systems close to a FM cause frustration and favor exotic states. There are theoretical between FM and AFM correlations in this system. This might that the field dependence of the magnetization and the properties. While some aspects of the theory agree very well with the experimental results (see, e.g., the fits in Fig. 29), there also are discrepancies. For instance, the theory predicts that the spin-freezing temperature $T_g$ from Lausberg et al., 2012.

A summary of the $\mu$SR results in zero and small longitudinal field is shown in Fig. 36. In conclusion, there is no FM QCP in CeFePO but rather some short-range order with a particular texture whose nature is unknown. The fact that in CeFePO strong FM fluctuations are present, and that AFM phases were observed in samples slightly doped with Ru or As (see Secs. II.D.2.c and II.D.2.b), suggests a delicate interplay between FM and AFM correlations in this system. This might cause frustration and favor exotic states. There are theoretical predictions of textured states in itinerant systems close to a FM instability; see Sec. III.E. An interesting proposal was put forward by Thomson, Krüger, and Green (2013), who suggested a helical glass state as the result of weak disorder which destabilizes the FM state and leads to incommensurate spiral magnetic order. Even the best samples of CeFePO have a relatively small RRR of approximately 5, but this may be attributed to the presence of strong FM fluctuations rather than quenched disorder. Since the amount of disorder necessary to generate the spiral state is rather small, the mechanism proposed by Thomson, Krüger, and Green (2013) is a viable candidate for explaining the observations.

5. Discussion and comparison with theory

The properties of CePd$_{1-x}$Rh$_x$ and Ni$_{1-x}$V$_x$ have been interpreted in terms of the theoretical ideas discussed in Secs. III.D.1 and III.D.3, namely, a quantum Griffiths phase and interactions between rare regions that lead to glassy properties. While some aspects of the theory agree very well with the experimental results (see, e.g., the fits in Fig. 29), there also are discrepancies. For instance, the theory predicts that the field dependence of the magnetization and the temperature dependence of the specific-heat coefficient are governed by the same exponent $\lambda$, whereas the data yield different values; see Fig. 29. For Kondo systems there are other scenarios that so far have not been explored in the context of these experiments; see Sec. III.D.2. The evidence for a glassy phase in UNi$_{1-x}$Co$_x$Si$_2$ (Sec. II.E.1.d) is much weaker, and the lowest $T_c$ where a clear FM transition has been observed is $T_c = 8.6$ K. Experiments at lower temperatures in the region close to $x = 1$ are needed to determine the nature of the FM QCP in this system.

The evidence for strong-disorder effects in the systems discussed in Sec. II.E.2 is weaker and largely based on the shape of the phase diagram. As discussed in Sec. III.B.3 there are other possible explanations for a tail in the phase diagram (see Fig. 38) and further investigations are needed to ascertain whether quantum Griffiths or related effects are indeed present in these materials.

III. THEORETICAL RESULTS

A. Soft modes in metals

In Sec. I.B we discussed why, in the absence of soft modes other than the order-parameter fluctuations, the quantum FM transition in clean systems is expected to be generically continuous with mean-field static exponents and a dynamical exponent $z = 3$. We also mentioned that this expectation breaks down in general, and in Sec. II we discussed many experiments that show a first-order transition rather than a continuous one. Indeed, Tables I, II, III, IV, and V show that most of the observed low-temperature transitions into a homogeneous FM phase are first order unless the system is either disordered or quasi-1D. We argue next that the systems in Table III can be understood in terms of a crossover from an asymptotic first-order transition to a preasymptotic regime that is described by Hertz theory. Underlying the breakdown of the original expectation for metallic magnets are soft two-particle excitations that couple to the magnetization. These excitations are the result of two characteristic properties of metals, viz., (1) a sharp Fermi surface, and (2) a nonzero density of states at the Fermi level. As we will see, they can be understood as representing a Goldstone mode that results from a spontaneously broken gauge symmetry. In disordered systems, soft critical order-parameter fluctuations exist in addition to this fermionic soft mode and govern the critical behavior. Finally, the rare-region effects mentioned in Sec. I.B can also be understood as a certain class of soft excitations. In this section we explain the importance of soft modes, give a classification, and discuss in more detail the fermionic Goldstone mode that is of particular importance for quantum FMs.

1. Why we should care about soft modes

The only way in which observables, be it thermodynamic quantities or time-correlation functions, can display nonanalytic behavior as a function of temperature, frequency, wave number, or an external field is via the existence of soft modes, i.e., correlation functions that diverge as the frequency and the wave number go to zero. It is useful in this context to consider soft modes as leading to a distribution of
relaxation times. Consider, for example, a diffusive process [see, e.g., Forster (1975)], i.e., a correlation function that behaves for small wave vectors $k$ and long times $t$ as

$$C(k, t) \propto e^{-i k z^2}$$

with $\nu$ some kinetic coefficient. This corresponds to a $k$-dependent relaxation time

$$\tau(k) = 1/\nu k^2.$$  \hspace{1cm} (3.1b)

For a fixed wave vector the decay is exponential, but the relaxation time diverges as $k \to 0$. As a result, the local time-correlation function $C(x = 0, t)$ decays algebraically with time. We write the local time-correlation function as an integral,

$$C(x = 0, t) = \int_0^\infty d\tau P(\tau) e^{-i \tau t}.$$  \hspace{1cm} (3.2a)

Here $\tau$ is a relaxation time $\tau_0 = 1/\nu k_0^2$ with $k_0$ the upper cutoff on the momentum integral in Eq. (3.2), and $P(\tau)$ is a relaxation-time distribution function. Comparing Eqs. (3.2) and (3.3a) we see that

$$P(\tau) \propto 1/t^{(d+2)/2}.$$  \hspace{1cm} (3.3b)

The algebraic decay, or long-time tail, of the time-correlation function is thus a result of the power-law decay of the distribution function $P(\tau)$. The Laplace transform $C(z)$ of $C(x = 0, t)$ is a nonanalytic function of the complex frequency $z$ at $z = 0$; for $\text{Im} z > 0$ and $(d-2)/2$ not integer the leading low-frequency behavior is

$$C(z) = i \int_0^\infty d\tau e^{iz\tau} C(x = 0, t) \propto z^{(d-2)/2} + \text{terms analytic in } z.$$  \hspace{1cm} (3.4)

Observables that couple to such diffusive modes will be given in terms of integrals whose integrands contain diffusive correlation functions. This results in a nonanalytic dependence on, e.g., the temperature or the frequency.

Such nonanalytic behavior can be generic, i.e., exist in an entire phase, or it can occur only at special points in the phase diagram. An example of the former is Goldstone modes, which arise from the spontaneous breaking of a global continuous symmetry (Forster, 1975; Zinn-Justin, 1996). Prime examples of the latter are critical fluctuations, which are soft only at a critical point (Ma, 1976). Other mechanisms for producing generic soft modes include conservation laws and gauge invariance (Belitz, Kirkpatrick, and Vojta, 2005). For the purposes of this review, we are interested in four classes of soft modes in metals. The first class consists of

(i) Single-particle excitations: These are represented by the soft single-particle Green’s function. They exist because of the sharpness of the Fermi surface and the existence of a nonvanishing quasiparticle weight. They are responsible for the leading behavior of observables in a Fermi liquid, e.g., the linear $T$ dependence of the specific heat.

An example of the effects of soft single-particle excitations is the paramagnon propagator in a metallic magnet. As a function of the wave vector $k$ and the imaginary Matsubara frequency $i\Omega$ it has the form (Doniach and Engelsberg, 1966; Hertz, 1976)

$$P(k, i\Omega) = \frac{1}{t + a k^2 + b |\Omega|^n/|k|^n}.$$  \hspace{1cm} (3.5a)

Here $t$ is the distance from the magnetic transition, and $a$ and $b$ are constants. $n$ is an integer that depends on the physical situation. For clean and disordered metallic FMs, $n = 1$ and $2$, respectively. For AFMs, $n = 0$. The spectrum or dissipative part of the corresponding causal function $P''(k, \omega) = \text{Im} P(k, i\Omega \to \omega + i0)$ reads

$$P''(k, \omega) = \frac{\omega |k|^n}{(t + a k^2)^2 + \omega^2 |k|^{2n}}.$$  \hspace{1cm} (3.5b)

We see that the paramagnon excitation is damped with a damping coefficient given by $\omega/|k|^n$. Physically, this reflects the coupling of the magnetic collective mode to the soft single-particle excitations in the itinerant electron system. It is usually referred to as Landau damping, in analogy to the corresponding effect in a collisionless classical plasma (Lifshitz and Pitaevskii, 1981). The same damping mechanism is applicable to the plasmon mode in a charged Fermi liquid, and the zero-sound mode in a neutral one (Pines and Nozières, 1989). We also note that the $|\Omega|$ singularity on the imaginary-frequency axis in Eq. (3.5) implies, for fixed $k$, a $1/t^2$ long-time tail for $P$ as a function of the imaginary time $\tau$ [see, e.g., Belitz, Kirkpatrick, and Vojta (2005)]. We will encounter the FM paramagnon propagator again later in this section [see Eqs. (3.44) and (3.75)], and the $1/t^2$ long-time tail will be important for the discussion in Sec. III.D.3.

The second class consists of

(ii) Two-particle excitations that are the Goldstone modes of a broken gauge symmetry with the density of states at the Fermi level as the order parameter. They were first identified as Goldstone modes by Wegner (1979) in the context of disordered electron system; we will discuss them in Sec. III.A.2.

The third class is

(iii) Griffiths or rare-region effects in disordered systems (Griffiths, 1969; McCoy, 1969).

These are normally not thought of as soft modes. To see the connection, consider a classical Ising system with randomly missing bonds in its disordered phase. In an infinite system, below the transition temperature of the clean system, but above the one of the actual bond-disordered one, one can find arbitrarily large regions with linear dimension $L$.
that happen to contain no missing bonds. In such a region, the spins are ordered, but the probability of finding such a region will decrease exponentially with its volume $L^d$. In order to destroy such a rare region, a surface free energy must be overcome. The relaxation time for a cluster of linear size $L$ will therefore be (Randeria, Sethna, and Palmer, 1985; Bray, 1988)

$$\tau(L) = \tau_0 e^{\sigma L^{d-1}}, \quad (3.6)$$

with $\tau_0$ a microscopic time scale and $\sigma$ a surface tension. This time scale diverges as $L \to \infty$, just as the diffusive relaxation time in Eq. (3.1b) diverges as $k \to 0$, only here the divergence is exponential. In order to estimate time-correlation functions $C(t)$ we need to weigh a factor $\exp[-t/\tau(L)]$ with the probability $P(L) \propto \exp(-cL^d)$ of finding a rare region in the first place and integrate over all sizes $L$. We thus expect

$$C(t) \propto \int_0^\infty dL \exp[-cL^d - (t/\tau_0)e^{-\sigma L^{d-1}}], \quad (3.7a)$$

where $c$ is a constant. For large times $t$ the integral can be evaluated by the method of steepest descent. The typical length scale $L$ is $L_{\text{typ}} \propto [\ln(t/\tau_0)]^{1/(d-1)}$, and the leading contribution to $C(t)$ is (Randeria, Sethna, and Palmer, 1985)

$$C(t \to \infty) \propto \exp\{-b[\ln(t/\tau_0)]^{d/(d-1)}\}, \quad (3.7b)$$

with $b = c/\sigma^{d/(d-1)}$ another constant. We see that the time-correlation function again decays slower than exponentially, albeit faster than any power. We again define a distribution function for relaxation times by writing

$$C(t) = \int_{\tau_0}^\infty dt P(\tau)e^{-t/\tau}. \quad (3.8a)$$

The leading behavior for large $\tau$ is

$$P(\tau \to \infty) \propto \exp\{-b[\ln(\tau/\tau_0)]^{d/(d-1)}\}. \quad (3.8b)$$

The analogy to diffusive soft modes, Eqs. (3.1)–(3.3), is now obvious.

For later reference we mention that Eq. (3.6) holds for classical Ising magnets only; for other systems the exponent may be different. We will discuss Griffiths effects in more detail in Sec. III.D.

All of these are generic soft modes. In addition, we will encounter

(iv) Critical fluctuations at the QCP in disordered FMs.

These are analogous to the critical fluctuations at classical transitions (Ma, 1976). However, as seen in Sec. III.C.2 their effects at the quantum ferromagnetic transition are rather weak.

Another class of generic soft modes is represented by phonons or elastic deformations in a continuum model. Their coupling to the magnetization has been studied extensively for classical magnets [see Bergman and Halperin (1976), and references therein]. For quantum magnets no convincing treatment exists; see Sec. III.F.

2. Goldstone modes in metals

We now illustrate the nature of the listed second class of soft modes, which are of crucial importance for the breakdown of Hertz theory. We first show that spinless noninteracting electrons, at $T = 0$, possess Goldstone modes resulting from a spontaneously broken gauge symmetry. We then generalize these arguments to the case of interacting electrons with spin.

a. Goldstone modes in a Fermi gas

Consider free electrons with mass $m_e$ and chemical potential $\mu$ described by fermionic (i.e., Grassmann-valued) fields $\bar{\psi}_n(k)$ and $\psi_n(k)$ that depend on a wave vector $k$ and a fermionic Matsubara frequency $\omega_n = 2\pi T(n+1/2)$ ($n = 0, \pm 1, \pm 2, \ldots$). These fields are temporal Fourier transforms of fields $\bar{\psi}(k, \tau)$ and $\psi(k, \tau)$ that depend on the imaginary-time variable $\tau$. In terms of these fields, the free-fermion action reads (Negele and Orland, 1988)

$$S_0[\bar{\psi}, \psi] = \sum_k \sum_n \bar{\psi}_n(k)[i\omega_n - k^2/2m_e + \mu]\psi_n(k). \quad (3.9)$$

Single-particle excitations are described by the Green’s function

$$G_n(k) = \langle \psi_n(k)\bar{\psi}_n(k) \rangle = 1/(i\omega_n - \xi_k) \quad (3.10)$$

with $\xi_k = k^2/2m_e - \mu$. These are soft in the sense that $G_n(k)$ diverges for wave vectors on the Fermi surface $\xi_k = 0$ as the frequency approaches zero. Of greater interest in the current context are two-particle excitations. Consider the correlation function

$$D_{nm}(k, q) \equiv \langle \bar{\psi}_n(k+)\psi_m(k+)\bar{\psi}_m(k-)\psi_n(k-) \rangle = \delta_{nm}\delta_{q0}[G_n(k)^2 - G_n(k+)G_n(k-)], \quad (3.11a)$$

where $k_\pm = k \pm q/2$, and the second line follows from Wick’s theorem. Multiplying Eq. (3.11a) with the inverse of $G_n(k_+)$ and $G_m(k_-)$, respectively, and subtracting the resulting two equations, we find

$$(i\Omega_{n-m} - k \cdot q/m_e)D_{nm}(k, q) = G_n(k_+) - G_m(k_-). \quad (3.11b)$$

Now analytically continue to real frequencies according to $i\omega_n \to \Omega + i\delta, i\omega_m \to -\Omega - i\delta$, and consider the limit $q \to 0, \Omega \to 0$. Equation (3.11b) then becomes

$$D^{+-}(k, q \to 0; \Omega \to 0) \equiv (iG''(k, \Omega = 0)) = \frac{iG''(k, \Omega = 0)}{\Omega - k \cdot q/m_e}. \quad (3.12)$$

Here $D^{+-}(\Omega)$ is the analytic continuation of $D_{nm}$, and $G''$ denotes the spectrum of the Green’s function. $D^{+-}$ diverges in the limit of zero wave vector $q$ and zero frequency $\Omega$, provided the spectrum of the Green’s function is nonzero. For free electrons this is the case for all values of $k$; if we replace the free electrons by band electrons $\xi_k = \epsilon_k - \mu$, where $\epsilon_k$ is determined by the lattice structure, it remains true everywhere within the band. Equivalently, it is true whenever the density
of states is nonzero. We have thus identified the correlation function $D_{\text{nm}}(\mathbf{k}, q)$, Eq. (3.11a), as a soft mode of non-interacting electrons. The nature of this soft mode is ballistic, i.e., the frequency $\Omega$ scales linearly with the wave number.

This simple result is more general and significant than one might expect, as can be seen from the following analogy. Consider a classical XY ferromagnet with magnetization $\mathbf{m}$ in the presence of a small magnetic field $\mathbf{h}$. Let the magnitudes of $\mathbf{m}$ and $\mathbf{h}$ be $m$ and $h$, respectively. In the PM phase, $\mathbf{m}$ is proportional to $\mathbf{h}$, and $m(h) = \mathbf{h}$ is an analytic function of $\mathbf{h}$; in particular, $m(h = 0) = 0$. However, in the FM phase this is not true. $\mathbf{m}$ still points in the same direction as $\mathbf{h}$, but $m$ is not an analytic function of $h$ at $h = 0$: $m(h = \pm 0) = \pm m_0$, with $m_0$ the spontaneous magnetization. Now let the system be in the FM phase and consider an infinitesimal rotation of the field $\mathbf{h} \rightarrow \mathbf{h} + \Delta \mathbf{h}$ that leaves $\mathbf{h}$ unchanged. Then the magnetization simply follows the field, with $\mathbf{m}$ also unchanged. Hence $|\Delta \mathbf{m}|/m = |\Delta \mathbf{h}|/h$. But $|\Delta \mathbf{m}|/|\Delta \mathbf{h}|$ is the homogeneous transverse susceptibility $\chi_\perp$, and hence

$$h_{\chi_\perp} = m. \quad (3.13)$$

This simple argument (Ma, 1976) shows that the transverse susceptibility diverges in the limit of zero field everywhere in the ordered phase where $m(h \rightarrow 0) \neq 0$. It can be made technically more elaborate by proving a Ward identity that takes the form of Eq. (3.13) (Zinn-Justin, 1996), but the simple argument contains all physically relevant points: The soft mode (that is, the magnon or transverse magnetization fluctuation) is a Goldstone mode that results from a spontaneously broken continuous symmetry (Goldstone, 1961; Forster, 1975; Zinn-Justin, 1996); in this case, the rotational symmetry in spin space leads to a nonzero order parameter $m$.

Now return to free fermions. Consider a local (in imaginary time) gauge transformation

$$\bar{\psi}(\mathbf{k}, \tau) \rightarrow e^{-i\alpha \tau} \bar{\psi}(\mathbf{k}, \tau), \quad \psi(\mathbf{k}, \tau) \rightarrow e^{i\alpha \tau} \psi(\mathbf{k}, \tau) \quad (3.14a)$$

with $\alpha$ real, or, equivalently,

$$\bar{\psi}_n(\mathbf{k}) \rightarrow \bar{\psi}_{n-a}(\mathbf{k}), \quad \psi_n(\mathbf{k}) \rightarrow \psi_{n-a}(\mathbf{k}). \quad (3.14b)$$

The second and third terms in Eq. (3.9) are invariant under this transformation, but the frequency term is not; it acts analogously to a magnetic field in the classical XY model. Explicitly, we have

$$S_0[\bar{\psi}, \psi] \rightarrow \sum_k \sum_n \bar{\psi}_n(\mathbf{k})[i\omega_n - \mathbf{k}^2/2m_e + \mu + i\alpha] \psi_n(\mathbf{k}). \quad (3.15)$$

If we now let $\alpha \rightarrow 0$ and consider the expectation value $\langle \psi_{n\uparrow}(\mathbf{k}) \bar{\psi}_{m\downarrow}(\mathbf{k}) \rangle$ we see that, upon analytic continuation to real frequencies, $\alpha > 0$ vs $\alpha < 0$ makes the difference between a retarded and an advanced Green’s function.

The latter are not the same anywhere within the band, and the $U(1)$ gauge symmetry expressed by Eqs. (3.14) is thus spontaneously broken. Equation (3.12) can now be interpreted in analogy to Eq. (3.13). The spectrum of the Green’s function is the order parameter of a spontaneously broken continuous symmetry, the frequency acts as the field conjugate to the order parameter, and the soft correlation function $D^{++}$ is the Goldstone mode associated with the broken symmetry. This remarkable analogy was first found by Wegner (1979) [see also Schäfer and Wegner (1980) and McKane and Stone (1981)] in the context of disordered electrons, where the soft modes are diffusive and commonly referred to as “diffusons” (Akkermans and Montambaux, 2011). The same argument holds for clean electrons, with the diffusive soft modes replaced by ballistic ones (Belitz and Kirkpatrick, 1997, 2012). In these papers the symmetry considered was an $SO(2)$ rotation in frequency space that is isomorphic to the $U(1)$ gauge transformation above. We stress that the broken symmetry discussed previously, and the resulting existence of the soft modes, has nothing to do with the conservation law for the particle number.

**b. Goldstone modes in a Fermi liquid**

We now take into account electron-electron interactions and spin. Interactions have two effects. One is the appearance of an inelastic scattering rate, both in the Green’s function and in the propagator $D^{+\pm}$. However, this rate vanishes at $T = 0$. The second change is the appearance of an additional term on the right-hand side of Eq. (3.11b), which is related to a three-particle correlation function. This term has a different functional dependence on the interaction than the difference of the Green’s functions in Eq. (3.11b) and therefore cannot change the fact that $D^{+\pm}$ diverges in the limit of vanishing frequency and wave number (Belitz and Kirkpatrick, 2012). This is consistent with Fermi-liquid theory, which posits that there is a one-to-one correspondence between free-electron states and states in a Fermi liquid (Lifshitz and Pitaevskii, 1991). Basic properties such as the soft-mode spectrum will thus not be changed by interactions, and only the coefficients in the soft propagator will acquire interaction dependences.

Spin provides another complication, which is conveniently dealt with by means of introducing bosonic matrix variables $Q$:

$$Q_{nm}(x, y) \equiv \frac{1}{2} \left( \begin{array}{cccc} -\psi_{n\uparrow}(x)\psi_{m\downarrow}(y) & -\psi_{n\uparrow}(x)\psi_{m\downarrow}(y) & -\psi_{n\uparrow}(x)\psi_{m\downarrow}(y) & \psi_{n\uparrow}(x)\psi_{m\downarrow}(y) \\ -\psi_{n\downarrow}(x)\psi_{m\uparrow}(y) & -\psi_{n\downarrow}(x)\psi_{m\uparrow}(y) & -\psi_{n\downarrow}(x)\psi_{m\uparrow}(y) & \psi_{n\downarrow}(x)\psi_{m\uparrow}(y) \\ -\bar{\psi}_{n\downarrow}(x)\bar{\psi}_{m\uparrow}(y) & -\bar{\psi}_{n\downarrow}(x)\bar{\psi}_{m\uparrow}(y) & -\bar{\psi}_{n\downarrow}(x)\bar{\psi}_{m\uparrow}(y) & \bar{\psi}_{n\downarrow}(x)\bar{\psi}_{m\uparrow}(y) \\ -\bar{\psi}_{n\uparrow}(x)\bar{\psi}_{m\downarrow}(y) & -\bar{\psi}_{n\uparrow}(x)\bar{\psi}_{m\downarrow}(y) & -\bar{\psi}_{n\uparrow}(x)\bar{\psi}_{m\downarrow}(y) & \bar{\psi}_{n\uparrow}(x)\bar{\psi}_{m\downarrow}(y) \end{array} \right). \quad (3.16)$$
Here $\cong$ means that $Q$ is isomorphic to $\bar{\psi}\psi$. We also define the Fourier transforms

\[ Q_{n,m}(k,p) = \frac{1}{V} \int dx dy e^{-ikx+\bar{\psi}\sigma\psi} Q_{n,m}(x,y) \]  

(3.17a)

and

\[ Q_{n,m}(k;q) = Q_{n,m}(k + q/2,k - q/2). \]  

(3.17b)

The $4 \times 4$ matrix $Q_{n,m}$ can be expanded in a spin-quaternion basis

\[ Q_{n,m}(x,y) = \sum_{i=0}^{3} (\tau_i \otimes s_i) \bar{Q}_{n,m}(x,y), \]  

(3.18)

where $\tau_0 = s_0 = \mathbb{1}_2$ is the unit $2 \times 2$ matrix, and $\tau_{1,2,3} = -s_{1,2,3} = i\sigma_{1,2,3}$ with $\sigma_{1,2,3}$ the Pauli matrices. Inspection shows that $i = 0$ and $i = 1, 2, 3$ represent the spin-singlet and triplet channels, respectively. Similarly, $r = 0, 3$ represents the particle-hole channel, i.e., products of the form $\bar{\psi}\psi$, whereas $r = 1, 2$ represents the particle-particle channel, i.e., products of the form $\bar{\psi}\bar{\psi}$ or $\psi\psi$. We need only the particle-hole degrees of freedom. All of the $Q_{n,m}$ are not independent; a convenient choice of the independent elements are those with $n \geq m$.

These considerations show that the matrix elements $0_{n,m}Q_{n,m}$ with $n \geq 0$ and $m < 0$ are soft modes. It is easy to see, by using discrete symmetries that connect the various channels, that in the absence of external fields all of the $iQ_{n,m}$ with $n \geq 0, m < 0$ are soft (Belitz and Kirkpatrick, 1997, 2012). Symmetry-breaking fields change this. For instance, an external magnetic field gives a mass to the particle-particle channel, and also to two of the particle-hole spin-triplet channels ($i = 1, 2$ for a magnetic field in the $z$ direction). A nonzero magnetization with a homogeneous component in a magnetically ordered phase has the same effect.

To summarize, of the two-particle degrees of freedom $iQ_{n,m}(k;q)$ defined by Eqs. (3.16)–(3.18), those with $n \geq 0$ and $m < 0$ are soft modes in the sense that their two-point correlation functions diverge in the limit of vanishing wave vector $q$ and vanishing frequency $i\Omega_{n-m} \rightarrow \Omega + i0$. They represent the Goldstone mode of a spontaneously broken continuous symmetry expressed by the gauge transformation in Eqs. (3.14). Physically, the broken symmetry reflects the difference between retarded and advanced degrees of freedom, and the spectrum of the single-particle Green’s function is the corresponding order parameter. Note that the $iQ_{n,m}(k;q)$ are soft for any value of $k$ for which the spectrum of the Green’s function is nonzero. Thus there are an infinite number of soft two-particle modes in a Fermi liquid. This is qualitatively different from the case of electrons in the presence of quenched disorder (Wegner, 1979; Schäfer and Wegner, 1980; McKane and Stone, 1981; Pruisken and Schäfer, 1982), and it was instrumental for Wegner’s matrix nonlinear sigma model describing the Anderson metal-insulator transition (Wegner, 1979). The derivation of Schäfer and Wegner (1980) was later generalized to the case of interacting electrons in the presence of disorder (Belitz and Kirkpatrick, 1997). In the notation of Eq. (3.11b), the two crucial differences in the disordered case are (1) Only the zeroth moment with respect to $k$ of the disorder average of $Q_{n,m}, \sum_{k}Q_{n,m}(k,q)$, is soft if $n$ and $m$ have different signs, and (2) the resulting soft modes have a diffusive character, $\Omega \sim q^2$, as opposed to the ballistic modes in the clean case. Denoting the soft modes analogous to Eq. (3.12) by $D^{n,m}(q,\Omega)$, one has

\[ D^{n,m}(q \to 0, \Omega \to 0) = \frac{n(\epsilon_F)}{\Omega + D q^2}, \]  

(3.19)

with $n(\epsilon_F)$ the density of states at the Fermi level, and $D$ a diffusion coefficient. These diffusive soft modes are often referred to as diffusons in the literature, and their counterparts are referred to as Cooperons. In the language of the $Q$ matrices, Eqs. (3.16)–(3.18), the diffusons and Cooperons are given by the correlation functions of the $\sum_{k}Q_{n,m}(k;q)$. Note that there are many more soft modes in a clean system than in a disordered one, which makes the soft-mode analysis in clean systems more complicated.

### B. Effects of fermionic soft modes: Simple physical arguments

We now give simple arguments for why fermionic fluctuations cause the FM QPT in clean 2D or 3D metals to always be discontinuous. We then discuss how the presence of quenched disorder modifies this conclusion.

1. **Renormalized Landau theory**

We need a theory that describes the magnetization or order-parameter (OP) field $m$, the fermionic degrees of freedom or conduction electrons described by the Grassmann fields of Sec. III.A, and the coupling between them. Accordingly, the action consist of three parts,

\[ S[m;\bar{\psi},\psi] = -A_{\text{OP}}[m] + S_{\text{F}}[\bar{\psi},\psi] + S_{m}[m;\bar{\psi},\psi]. \]  

(3.20)

They denote a purely bosonic part of the action that governs the OP, a purely fermionic one that describes the conduction electrons, and a coupling between the two.\(^{20}\) The partition function is given by

\[ Z = \int D[m]D[\bar{\psi},\psi]e^{iS[m;\bar{\psi},\psi]}. \]  

(3.21)

\(^{20}\)We denote actions that depend on bosonic (number-valued) fields only by $A$, actions that depend on fermionic (Grassmann-valued) fields, or a mix of bosonic and fermionic fields, by $S$, and use a sign convention such that $S$ and $A$ enter the exponential with a plus and minus sign, respectively.
Note that we do not specify the origin of the magnetization; in general it can be due to the conduction electrons, or due to localized electrons in a different band, or a combination of the two. If one formally integrates out the conduction electrons one obtains an effective LGW action in terms of the OP only,

$$Z = \int D[m]e^{-A_{\text{eff}}[m]},$$

where

$$A_{\text{eff}}[m] = A_{\text{OP}}[m] - \ln \int D[\tilde{\psi}, \psi]e^{S_0[\tilde{\psi}, \psi]+S_0[m, \tilde{\psi}, \psi]}.$$  \hspace{1cm} (3.22b)

In general, the magnetization will couple both to the orbital angular momentum and to the spin of the electrons. The former poses interesting questions that have received little attention to date, and we do not discuss it here. The latter coupling is via a Zeeman-like term

$$S_0[m; \tilde{\psi}, \psi] = c \int dx m(x) \cdot n_s(x).$$  \hspace{1cm} (3.23)

Here $c$ is a coupling constant and $n_s$ is the electronic spin density,

$$n_s(x) = \sum_{a, b} \tilde{\psi}_a(x) \sigma_{ab} \psi_b(x),$$

with $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ the Pauli matrices, $a, b = (\uparrow, \downarrow)$ are spin indices, $x \equiv (x, \tau)$ comprises both the real-space position $x$ and the imaginary-time variable $\tau$, and $\int dx \equiv \int d\tau \int_0^T d\tau$, with $V$ the system volume.

For simplicity we now treat the OP in a mean-field approximation, i.e., we replace the fluctuating magnetization $m(x)$ by an $x$-independent magnetization $m$ that we take to point in the 3-direction. We will discuss the validity of this approximation in Sec. III.C. Denoting the 3-component of $n_s$ by $n_s$, the second term in Eq. (3.22b), which describes the effect of the coupling between the fermions and the OP, can be written

$$\delta A[m] = -\ln \int D[\tilde{\psi}, \psi]e^{S_0[\tilde{\psi}, \psi]+cm \int dx n_s(x)}$$

$$= -\ln \langle e^{cm \int dx n_s(x)} \rangle_T,$$  \hspace{1cm} (3.25)

where in the second line we dropped a constant contribution to the action, and $\langle \cdots \rangle_T$ denotes an average with the action $S_T$.

Now consider the longitudinal spin susceptibility $\chi(h)$ of fermions governed by the action $S_F$ and subject to a magnetic field $h$. It is given by the correlation function

$$\chi(h) = \frac{T}{V} \int dxdy \langle \delta n_s(x) \delta n_s(y) \rangle_{S_F},$$

where $S_T = S_F + h \int dx n_s(x)$, and $\delta n_s(x) = n_s(x) - \langle n_s(x) \rangle_{S_F}$. By differentiating Eq. (3.25) twice with respect to $m$ it is easy to show that

$$\frac{d^2 \delta A}{dm^2} = -\frac{V}{T} \chi'(cm).$$  \hspace{1cm} (3.27)

Since $\delta A[m = 0] = d\delta A/dm|_{m=0} = 0$, we now have

$$\delta A[m] = -\frac{V}{T} c^2 \int_0^m dm_1 \int_0^{m_1} dm_2 \chi'(cm_2).$$  \hspace{1cm} (3.28)

$S_{\text{OP}}$ will have the usual Landau form of a power series in powers of $m^2$, and the complete renormalized Landau free-energy density $f_{\text{eff}} = -(T/V)A_{\text{eff}}$ thus is

$$f_{\text{eff}}[m] = \frac{1}{2} m^2 + \delta f[m] + um^4 + O(m^6).$$  \hspace{1cm} (3.29a)

Here $t$ and $u$ are Landau parameters, and

$$\delta f[m] = -c^2 \int_0^m dm_1 \int_0^{m_1} dm_2 \chi'(cm_2).$$  \hspace{1cm} (3.29b)

This result expresses the correction to the usual Landau action in terms of the spin susceptibility of nonmagnetic fermions in the presence of an effective homogeneous magnetic field given by $cm$. It is a “renormalized Landau theory” in the sense that it includes the effects of fluctuations extraneous to the OP fluctuations. The remaining question is the behavior of the susceptibility $\chi$ that represents these fluctuations for small $m$. As we will see, $\chi$ is not an analytic function of $m$ at $m = 0$.

2. Clean systems

Various observables in a Fermi liquid are nonanalytic functions of the temperature. For instance, the specific-heat coefficient has a $T^2 \ln T$ term (Baym and Pethick, 1991). The spin susceptibility in a 3D system has no such nonanalytic behavior (Carneiro and Pethick, 1977). However, this absence of a nonanalyticity was later shown to be accidental, and to pertain only to the $T$ dependence in $d = 3$. In dimensions $d \neq 3$ there is a $T^{d-1}$ nonanalyticity, and even in $d = 3$ at $T = 0$ the inhomogeneous spin susceptibility has a $k^2 \ln k$ wave-number dependence (Belitz, Kirkpatrick, and Vojta, 1997; Chitov and Mills, 2001; Galitski, Chubukov, and Das Sarma, 2005). This nonanalyticity is a direct consequence of the soft modes discussed in Sec. III.A. From scaling arguments one expects a corresponding nonanalyticity for the homogeneous susceptibility at $T = 0$ as a function of a magnetic field $h$, namely, $\chi(h) \propto \text{const} + h^{d-1}$ in generic dimensions, and $\chi(h) \propto \text{const} - h^2 \ln h$ in $d = 3$. These scaling arguments have been shown to be exact, as far as the exponent is concerned, by a RG treatment (Belitz and Kirkpatrick, 2014), and they are consistent with explicit perturbative calculations (Misawa, 1971; Barnea and Edwards, 1977; Betouras, Efremov, and Chubukov, 2005). The sign of the effect is universal and can be established as follows. Fluctuations suppress the tendency of a Fermi liquid to order ferromagnetically, and therefore the fluctuation correction to the bare zero-field susceptibility is negative, $\delta\chi(0) < 0$. A magnetic field suppresses the fluctuations, and therefore $\delta\chi(h) - \delta\chi(0) > 0$. This implies that the nonanalyticity in $\chi(h \to 0)$ has a positive sign.
$\chi(h \to 0) = \chi(0) + \begin{cases} a_d h^{d-1}, & \text{for } 1 < d < 3, \\ a_3 h^2 \ln(1/h), & \text{for } d = 3, \end{cases}$ (3.30)

where $a_d > 0$. For the renormalized Landau free-energy density, Eq. (3.29a), we thus obtain

$$f_{\text{cl}}[m] = -hm + tm^2 + um^4 - v_d \times \begin{cases} m^{d+1} + um^4 & (1 < d < 3), \\ m^4 \ln(1/m) & (d = 3). \end{cases}$$ (3.31)

Here $v_d \propto e^{d+1} > 0$, and we have added an external magnetic field $h$. For $d = 3$ this result was first derived by Belitz, Kirkpatrick, and Vojta (1999). The negative term in the free energy, which dominates the quartic term for all $d \leq 3$, necessarily leads to a first-order ferromagnetic transition. We stress that while this is a fluctuation-induced first-order quantum phase transition, the relevant fluctuations are not the OP fluctuations, but are fermionic in nature. For purposes of an analogy with the well-known classical fluctuation-induced first-order transitions (Halperin, Lubensky, and Ma, 1974), the latter play a role that is analogous to that of the vector potential in superconductors, or the director fluctuations at the nematic-smectic-A transition. An important difference, however, is that in these classical systems the OP fluctuations are below their upper critical dimension, which makes them strong enough to make the first-order transition weak and hard to observe at best, and destroy it altogether at worst (Anisimov et al., 1990).

In contrast, in the case of a quantum FM the OP fluctuations are above their upper critical dimension, so the first-order transition predicted by the renormalized Landau theory will be much more robust.

A nonzero temperature cuts off the magnetic field singularity (Betouras, Efremov, and Chubukov, 2005), and with increasing temperature the fluctuation-induced term in the free energy becomes less and less negative. Suppose the Landau parameter $t$ at $T = 0$ is a monotonically increasing function of, say, hydrostatic pressure $p$, and let $t(p = 0, T = 0) < 0$. Then there will be a QPT at some nonzero pressure $p_c$. As the transition temperature is increased from zero by lowering $p$, one expects a tricritical point in the phase diagram. Below the tricritical temperature the transition will be discontinuous due to the mechanism described earlier, while at higher temperatures it will be continuous. In the presence of an external magnetic field there appear surfaces of first-order transitions, or tricritical wings (Belitz, Kirkpatrick, and Rollbühler, 2005), and the phase diagram has the schematic structure shown in the rightmost panel in Fig. 37. The third law of thermodynamics, in conjunction with various Clapeyron-Clausius relations, puts constraints on the shape of the wings at low temperatures (Kirkpatrick and Belitz, 2015b). Most importantly, the wings must be perpendicular to the $T = 0$ plane, and they cannot be perpendicular to the zero-field plane. These constraints, as well as the overall wing structure, are in excellent agreement with experimentally observed phase diagrams; see, for instance, Figs. 1, 3, 6, and 10.

The theory also predicts a correlation between the tricritical temperature $T_{tc}$ and the magnetic moment $m_1$ just on the FM side of the first-order transition. In terms of the parameters in Eq. (3.31) one finds $T_{tc} = T_0 e^{-u/v}$ and $m_1 = m_0 e^{-1/2u/v}$, where $T_0$ and $m_0$ are microscopic temperature and magnetization scales, respectively (Belitz, Kirkpatrick, and Rollbühler, 2005). For given scales $T_0$ and $m_0$, which one expects to vary little within members of a given class of materials, the theory thus predicts that $T_{tc}$ is proportional to $m_1$. As pointed out in Sec. II.B.5, this is in good agreement with experiments.

3. Disordered systems

In the presence of quenched disorder the logic of these arguments remains intact, but important aspects change. First, the fermionic soft modes are diffusive in nature, rather than ballistic; see Sec. III.A.2.c. This slowing down of the electrons favors the tendency toward ferromagnetism, and the combined disorder and interaction fluctuations increase the bare susceptibility $\delta\chi(0) > 0$.22 A small magnetic field again suppresses the effect of the fluctuations, and the nonanalytic contribution to $\chi(h)$ therefore has a negative sign. Second, the changed nature of the fermionic soft modes leads to a different exponent, namely,

22The notion that quenched disorder favors ferromagnetic order is somewhat counterintuitive, given that in classical systems, long-range order is negatively affected by it (Cardy, 1996). Indeed, the ferromagnetic $T_c$ usually decreases with increasing disorder; see Sec. II. However, at sufficiently low temperature the diffusive motion of the electrons leads to an increase in the effective exchange interaction. The interplay between these two effects is discussed at the end of this section.
\[ \chi(h \to 0) = \chi(0) - \tilde{a}_d h^{(d-2)}/2, \quad \text{for } d > 2, \]  
\[ \text{with } \tilde{a}_d > 0. \] 
This expectation is borne out by explicit perturbative calculations (Alshuler, Aronov, and Zyuzin, 1983). Third, the notion of a disordered Fermi liquid breaks down for \( d \leq 2 \) due to localization effects (Lee and Ramakrishnan, 1985; Belitz and Kirkpatrick, 1994), so the only physical dimension where the current discussion applies is \( d = 3 \).

The renormalized Landau free-energy density in \( d = 3 \) now becomes (Belitz and Kirkpatrick, 1996; Kirkpatrick and Belitz, 1996)
\[ f_{\text{eff}}[m] = -hm + tm^2 + vm^{5/2} + um^4, \]  
with \( t > 0. \) We see that, for very general reasons, quenched disorder leads to a second-order or continuous transition, but the Landau theory for this transition is not standard because of the \( m^{5/2} \) term which leads to unusual critical exponents. In particular, the exponents \( \beta \) and \( \delta \) for the OP, and \( \gamma \) for the OP susceptibility, are
\[ \beta = 2, \quad \delta = 3/2, \quad \gamma = 1. \]  
Other critical exponents are discussed in Sec. III.C.

Equations (3.31) and (3.33) are valid for the extreme cases of ultraclean and strongly disordered systems, respectively. An equation of state that interpolates between the two was constructed by Sang, Belitz, and Kirkpatrick (2014); the schematic evolution of the phase diagram with increasing disorder is shown in Fig. 37. The theory allows one to distinguish between the three distinct disorder regimes, characterized by the residual resistivity \( \rho_0 \):

**Regime I (clean):** \( \rho_0 \lesssim \rho_0^{(1)} \). The transition at low temperature is first order, and there is a tricritical point in the phase diagram. The tricritical temperature decreases with increasing disorder.

**Regime II (intermediate):** \( \rho_0^{(1)} \lesssim \rho_0 \lesssim \rho_0^{(2)} \). The transition is second order down to \( T = 0 \). The critical behavior is mean field like, as predicted by Hertz theory, except extremely close to the critical point, where it crosses over to the exponents given by Eq. (3.34).

**Regime III (disordered):** \( \rho_0 \gtrsim \rho_0^{(2)} \). The transition is second order, and the critical exponents are given by Eq. (3.34). In this regime the quantum Griffiths effects discussed in Sec. III.D are expected to be present and to compete with the critical behavior.

A rough estimate (see footnote 10) for the boundaries between the three regimes yields \( \rho_0^{(1)} \approx 1 \) to several \( \mu \Omega \text{cm} \), and \( \rho_0^{(2)} \approx 100 \) to several hundred \( \mu \Omega \text{cm} \).

We now discuss the expected qualitative shape of the phase diagram. Let \( x \) be a dimensionless measure of the disorder, \( x \propto 1/\tau \) with \( \tau \) the elastic mean-free time. As mentioned previously (see footnote 22), there are two competing influences of \( x \) on the critical temperature. One is a classical dilution effect that suppresses \( T_c \) to zero at sufficiently large values of \( x \) (Cardy, 1996). For simplicity, let us assume that this leads to \( T_c(x) = 1 - x^2 \), with \( T_c \) measured in units of \( T_c(x = 0) \). (Adding a term linear in \( x \) does not change the qualitative discussion that follows.) The other is an increase in \( T_c \) due to the diffusive nature of the electron dynamics, which increases the effective spin-triplet interaction (Alshuler, Aronov, and Zyuzin, 1983). Indeed, the increase in the zero-field susceptibility mentioned before Eq. (3.32) is proportional to this increase in the interaction amplitude. For small disorder, this effect is linear in the disorder at \( T = 0 \), and it is cut off by the temperature itself, i.e., it is strongest for small values of \( T_c \). A simple schematic way to represent these two effects is
\[ T_c(x) = 1 - x^2 + \frac{ax}{1 + bT_c(x)/x}. \]  

The resulting qualitative shape of the phase diagram is shown in Fig. 38. The two competing disorder effects can lead to an inflection point in \( T_c(x) \) vs \( x \) that is often seen in experiments; see Figs. 28 and 33. Another possible interpretation of this shape of the phase diagram is a smeared transition due to quantum Griffiths effects that have been ignored in these arguments; this is discussed in Sec. III.D. In addition to the inflection point in the phase diagram, the unusually large value of the exponent \( \beta \), Eq. (3.34), can also mimic a smeared transition.

We finally recall a very general result for systems with quenched disorder due to Harris (1974) and Chayes et al. (1986). Harris investigated a necessary condition for the critical behavior of a clean system to remain unchanged by a small amount of quenched disorder. In order for the transition to stay sharp, the disorder-induced fluctuations of the location of the critical point in parameter space must be small compared to the distance from the critical point. This implies that the correlation length must diverge sufficiently fast as the critical point is approached, which leads to a requirement for the correlation-length exponent \( \nu \),
\[ \nu \geq 2/d. \]  
This is often referred to as the Harris criterion. In Harris’s original argument this was a constraint on the exponent \( \nu \) of the clean system which, if hyperscaling holds, is equivalent to the condition \( \alpha < 0 \) for the specific-heat exponent \( \alpha \). It does not apply to the quantum FM transition, since the latter is not
continuous in clean systems. However, Chayes et al. (1986) showed rigorously that, for a large class of disordered systems that undergoes a continuous phase transition, Eq. (3.36) must hold for the exponent \( \nu \) that characterizes the disordered fixed point. The value of \( \nu \) corresponding to the exponents given in Eq. (3.34), viz., \( \nu = 1 \) in \( d = 3 \) (Kirkpatrick and Belitz, 1996), satisfies this constraint, while the mean-field value \( \nu = 1/2 \) of Hertz theory does not. We come back to this point in Sec. III.C.2.

### C. Effects of order-parameter fluctuations, and comparison with experiment

In Sec. III.B.3 we treated the OP in a mean-field approximation and integrated out all fermionic degrees of freedom. The fermionic soft modes then led to a Landau free energy that is not an analytic function of the magnetization. We now discuss the order-parameter fluctuations that were neglected in this procedure and also consider the behavior not asymptotically close to the quantum phase transition. All of these issues are important for the relation of the theory to the experimental results discussed in Sec. II.

#### 1. Coupled field theory for soft modes

We return to the coupled field theory in Eq. (3.20). For the purposes of discussing fluctuation effects, integrating out the fermions, as in Eq. (3.22), is not desirable, as it results in a nonlocal field theory for the order-parameter fluctuations. A better strategy is to separate the fermionic degrees of freedom into soft and massive modes and integrate out only the latter to arrive at an effective theory that treats all of the soft modes on equal footing. This is possible using the identification of fermionic soft modes explained in Sec. III.A. The resulting effective action can then be analyzed by means of RG methods.

In Sec. III.A we have seen that the soft fermionic degrees of freedom are given by those matrix elements \( Q_{nm} \), Eq. (3.16), for which the two Matsubara frequencies have different signs. Denoting these by \( \Omega_{nm} \) with \( n > 0, m < 0 \) implied, see the remark after Eq. (3.10)], and the massive modes by \( P_{nm} \), we first rewrite the action from Eq. (3.20) in terms of the \( q \) and \( P \):

\[
\mathcal{A}[m; q, P] \equiv -S[m; \psi, \varphi] = \mathcal{A}_{\text{OP}}[m] + \mathcal{A}_F[q, P] + \mathcal{A}_c[m; q, P].
\]

If we now integrate out the massive modes \( P \), we can formally write the partition function

\[
Z = \int D[m] D[q] e^{-\mathcal{A}_{\text{OP}}[m; q]},
\]

in terms of an effective action

\[
\mathcal{A}_{\text{eff}}[m, q] = \mathcal{A}_{\text{OP}}[m] - \ln \int D[P] e^{-\mathcal{A}_F[q, P] - \mathcal{A}_c[m; q, P]} \equiv \mathcal{A}_{\text{OP}}[m] + \mathcal{A}_F[q] + \mathcal{A}_c[m, q].
\]

Integrating out the \( P \) cannot be done exactly, but any approximation that respects the symmetries of the action suffices.

Before we discuss the various terms in this effective action in more detail, we make a few general remarks. \( \mathcal{A}_{\text{OP}} \) is a standard LGW action, supplemented by a random-mass term in the disordered case; see Eq. (3.74) below. \( \mathcal{A}_F \) has a Gaussian contribution that reflects the soft modes identified in Sec. III.A.2, as well as higher-order terms to all orders in \( q \).

The soft modes are diffusive in disordered systems, and ballistic in clean ones, but apart from this and the random-mass term in \( \mathcal{A}_{\text{OP}} \) there are no structural differences between clean and disordered systems as far as these two terms in the action are concerned. The coupling \( \mathcal{A}_c \) has a contribution that is bilinear in \( m \) and \( q \), and in addition terms of order \( mq^2 \), where \( n \) can be any integer. The bilinear term leads to the characteristic Landau damping in the paramagnon propagator (Doniach and Engelsberg, 1966; Hertz, 1976), i.e., to a frequency dependence of the form \( \Omega/[k] \) in clean systems, and \( \Omega/k^2 \) in disordered ones. At the level of terms bilinear in \( m \) and \( q \) there is again no other structural difference between the clean and disordered cases. However, the terms of order \( mq^2 \) generate, in a renormalization procedure, a nonanalytic wave-number dependence of the paramagnon propagator that has the form \( |k|^{d-1} \) in clean systems, and \( |k|^{d-2} \) in disordered ones. The sign of this term is different in the two cases. If one replaces the fluctuating order parameter by its expectation value, this term leads to the renormalized Landau theory described in Sec. III.B.1.

#### 2. Clean systems

In clean systems, the transition at \( T = 0 \) in zero field is first order if OP fluctuations are neglected; see Sec. III.B.2. OP fluctuations are thus cut off before the system reaches a critical point, remain finite, and do not change the nature of the transition. However, if the transition is weakly first order there will be a sizable region in parameter space where the

23Historically, this was the route taken by Hertz (1976), who integrated out the fermions in a tree approximation. For disordered systems, it was later refined by Kirkpatrick and Belitz (1996), who showed that fermionic loops destabilize Hertz’s critical fixed point. While this method works for power-counting purposes, the coupled local field theory developed later for disordered (Belitz et al., 2001a, 2001b) and clean (Kirkpatrick and Belitz, 2012b) systems is more versatile and easier to handle, and we use it here. For clean systems, the fermionic fluctuations destroy the critical fixed point and change the order of the transition, as discussed in Sec. III.B.2. However, the preasymptotic behavior, which is governed by a critical fixed point that ultimately is unstable, can still be important and is discussed in Sec. III.C.2.

24Technically, this can be achieved by constraining all terms in the action that contain the Grassmann fields to higher than bilinear order to the matrix field \( Q \) from Eq. (3.16) by means of a Lagrange multiplier field and integrating out the Grassmann fields; see Belitz and Kirkpatrick (1997, 2012). For simplicity we suppress the dependence of the action on the Lagrange multiplier field in our notation.

25This needs to be interpreted with care in light of footnote 8. The first-order transition is described by a strong-coupling fixed point (Nienhuis and Nauenberg, 1975; Fisher and Berker, 1982), and the relevant fluctuations are effectively already included in the generalized Landau theory represented by Eq. (3.31).
physical behavior is controlled by the unstable fixed point that is described by Hertz theory, and only asymptotically close to the transition will the RG flow turn away toward the strong-coupling fixed point that describes the first-order transition. It is therefore important to fully understand the results of Hertz theory and its predecessors, even if they ultimately do not describe the nature of the transition correctly. Also, OP fluctuations affect the various lines of second-order transitions in the phase diagram shown schematically in Fig. 37.

a. Hertz’s action, and relation to spin-fluctuation theory

In clean systems, the relevant fermionic soft modes are the Fermi-liquid Goldstone modes discussed in Sec. III.A.2. The soft-mode action has not been derived in a closed form, but can be obtained to any desired order in the soft degrees of freedom $q$. To Gaussian order it reads (Belitz and Kirkpatrick, 2012)

$$ A_{V}[q] = -8\sum_{r=0}^{3} \sum_{i=0}^{3} \sum_{k \in \mathbb{Q}} \frac{1}{V} q_{12}(k)^{i} q_{34}(-k) \times \left( \delta_{13} \delta_{24} - \delta_{1-2,3-4} 2T \gamma_{i} \right) + O(q^{3}). \quad (3.39) $$

Here $\gamma_{0}$ and $\gamma_{1,2,3}$ are the spin-singlet and spin-triplet interaction amplitudes, respectively, $1 \equiv n_{1}$ etc. is a shorthand for Matsubara frequencies, and the function $\varphi$ is given by

$$ \varphi_{12}(k) = \frac{2\pi G}{k} \varphi_{d}(Gk \Omega_{1}^{-1} k), \quad (3.40) $$

where $G$ is a coupling constant whose bare value is the inverse Fermi velocity $1 / v_{F}$. $\varphi_{d}$ can be expressed in terms of Gauss’s hypergeometric function. For $d = 1, 2, 3$ it reduces to the familiar expressions for the hydrodynamic part of the Lindhard function in these dimensions:

$$ \varphi_{d=1}(z) = -iz/(1 - z^{2}), \quad (3.41a) $$

$$ \varphi_{d=2}(z) = i\sqrt{z + 1} \sqrt{z - 1}, \quad (3.41b) $$

$$ \varphi_{d=3}(z) = -i \frac{1}{2} \ln \left( \frac{1 + z}{1 - z} \right). \quad (3.41c) $$

Note that the vertex $1 / \varphi$ scales as a function that is linear in either the frequency or the wave number (except in $d = 1$), and that this is true also for the interaction term in Eq. (3.39) due to the structure of the frequency constraint. This feature reflects the Goldstone modes.

The order-parameter field $m$ couples linearly to the electron spin density with a dimensionless coupling constant $c = O(1)$. To linear order in the soft component $q$ the coupling reads

$$ A_{V}[m, q] = 8c\sqrt{T} \sum_{k \in \mathbb{Q}} \sum_{i,j=12} \frac{1}{12} q_{12}(k)^{i} b_{12}(-k) + O(qm^{2}), \quad (3.42a) $$

where

$$ i b_{12}(k) = (-)^{r / 2} \sum_{n} \delta_{n,n_{1} - n_{2}} [m_{n}^{i}(k) + (-)^{r + 1} m_{n}^{r + 1}(k)] \quad (3.42b) $$

is a symmetrized version of the order-parameter field $m_{n}(k)$ with components $m_{n}^{1,2,3}$.

Finally, the order-parameter action is an ordinary quantum $\phi^{4}$ theory,\(^{26}\)

$$ A_{\text{OP}}[m] = -\sum_{k,n} m_{n}(k) \left[ t + a k^{2} + b(\Omega_{n})^{2} \right] \cdot m_{-n}(-k) \quad + u \int d\Omega \sum_{n_{1},n_{2},n_{3}} [m_{n_{1}}(x) \cdot m_{n_{2}}(x)] \times \left[ m_{n_{3}}(x) \cdot m_{-n_{1} - n_{2} - n_{3}}(x) \right], \quad (3.43) $$

with $t, a, b, \quad$ and $u$ the coupling constants of this LGW action.

We now have specified all parts of the effective action, Eq. (3.38b), to bilinear order in $m$ and $q$. This is not a fixed-point action corresponding to a critical fixed point. The terms of $O(mq^{2})$ that are not shown explicitly in Eq. (3.42) are relevant with respect to the fixed point represented by this action and lead to the first-order transition described in Sec. III.B.2. However, depending on the strength of the first-order transition, there will be a sizable regime where the RG flow is dominated by the unstable fixed point. The physical behavior in this regime will thus be given by the action as written earlier, before it crosses over to the first-order transition. To study this preasymptotic behavior it is convenient to integrate out the fermion fields $q$, which yields the action derived by Hertz (1976). In particular, the Gaussian OP or paramagnon propagator reads

$$ \langle m_{n}^{i}(k) m_{m}^{j}(p) \rangle = \delta_{i,j} \delta_{n,-m} \frac{1}{2 \pi} \frac{1}{t + a k^{2} + b(\Omega_{n})^{2} + Gc|\Omega_{n}|/|k|}. \quad (3.44) $$

Here we replaced the vertex $1 / \varphi_{12}(k)$ in Eq. (3.39) with a schematic one that is linear in $\Omega$ and $k$ for simplicity. We see that the coupling to the electronic Goldstone modes generates the characteristic Landau-damping term proportional to $|\Omega_{n}|/|k|$ in the paramagnon propagator. The term quadratic in the frequency in Eq. (3.43) is therefore not the leading frequency dependence and can be dropped. The approximate effective action becomes

\(^{26}\)This action is missing a term of order $\Omega m^{4}$ that describes the Bloch spin precession of the OP in the field of all the other magnetic moments. This term is absent in the case of an Ising OP, but in all other cases it is important for producing the correct dynamics of the spin waves. In a field-theoretic context, it is sometimes referred to as a Wess-Zumino or Chern-Simons term, and its topological aspects are stressed (Fradkin, 1991). For the purposes of our discussion it is RG irrelevant, and we drop it.
\[ A_{\text{Hertz}} = -\sum_{\mathbf{k}, n} m_n(k) [t + ak^2 + Gc|\Omega_n|/|k|] \cdot m_{-n}(k) \]
\[ + u \int dxT \sum_{n_1,n_2,n_3} [m_{n_1}(x) \cdot m_{n_2}(x)] \]
\[ \times [m_{n_3}(x) \cdot m_{-n_1-n_2-n_3}(x)]. \tag{3.45} \]

This action was derived and studied by Hertz (1976), and its finite-temperature properties were analyzed by Millis (1993). Many of the explicit results were derived earlier by means of a theory of spin fluctuations that one would now classify as a self-consistent one-loop theory; see Moriya (1985), Lonzarich and Tennialer (1985), and Lonzarich (1997). This development was analogous to that in the area of classical critical dynamics, where mode-coupling theories (Fixman, 1962; Kawasaki, 1967, 1970, 1976; Kadanoff and Swift, 1968) were followed by RG treatments (Hohenberg and Halperin, 1977). In what follows, we will derive these results by means of scaling arguments, which is analogous to a third angle of attack on the classical dynamical scaling problem (Ferrell et al., 1967; Halperin and Hohenberg, 1967; Ferrell et al., 1968).

b. Scaling analysis of the preasymptotic regime

From the action, Eq. (3.45), we see that the frequency scales as \( \Omega \sim k^3 \). That is, the dynamical exponent is
\[ z = 3, \tag{3.46} \]

independent of the dimensionality. The theory thus is above its upper critical dimension for all \( d > 4 - z = 1 \). Let \( t \) be the distance from criticality at \( T = 0 \) and define static exponents by the dependence of the observables on \( t \) in the usual way; see Appendix B. The static exponents \( \nu, \beta, \eta, \gamma, \) and \( \delta \) then all have their usual mean-field values for all \( d > 1 \):

\[ \nu = 1/2, \quad \beta = 1 - 4 = 0, \quad \eta = 1, \quad \gamma = 3. \tag{3.47} \]

The quartic coefficient \( u \) with scale dimension \( u = -(d + z - 4) = -(d - 1) \) is a dangerous irrelevant variable (DIV) with respect to the order parameter, which is why \( \beta \) and \( \delta \) deviate from the naive scaling results. [For a general discussion of the DIV concept, see, Ma (1976) and Fisher (1983).]

The coefficient \( u \) also plays an important role for the temperature dependence of many observables, which is not simply determined by \( z \) due to the dangerous irrelevance of \( u \). We now show how the relevant results can be obtained from scaling arguments. Scaling functions will be denoted by \( F \) with a subscript indicating the observable in question.

(i) Correlation length: Let \( \delta(t, T) \) be the \( T \)-dependent distance from the QCP, such that \( \delta(t, 0) = t \). Then the homogeneity law for \( \delta \) is

\[ \delta(t, T) = b^{-7/4} F_\delta(tb^{1/4}, Tb^2, ub^{-(d-1)}) \]
\[ = b^{-2} F_\delta(tb^2, Tb^3, ub^{-(d-1)}). \tag{3.48} \]

The temperature dependence of \( \delta \) results from a one-loop contribution that is proportional to \( u \). Therefore,
\[ F_\delta(0, 1, y \to 0) \propto y. \tag{3.49a} \]
This is in agreement with the perturbative result for \( d = 3 \) obtained by Moriya and Kawabata (1973),
\[ \delta = t + \text{const} \times uT^{3/4}. \tag{3.49b} \]
\( u \) is thus dangerously irrelevant with respect to the \( T \)-dependence of \( \delta \). In contrast, \( F_\delta(1, 0, y \to 0) \propto \text{const} \) and also \( \partial F_\delta(x, 1, y \to 0)/\partial x \bigg|_{x=0} = \text{const} \). Choosing \( b = T^{-1/3} \) we can therefore Taylor expand in the first argument of \( F_\delta \). Specializing to \( d = 3 \) we have, for \( t/T^{2/3} \ll 1 \),
\[ \delta(t, T) = T^{2/3} F_\delta(t/T^{2/3}, 1, uT^{3/3}) \]
\[ \propto T^{2/3}[uT^{2/3} + \text{const} \times t/T^{2/3} + \cdots]. \tag{3.50} \]

For \( T_c \), defined by \( \delta(t, T_c) = 0 \), this gives \( t \propto T_c^{2/3} \times uT_c^{2/3} \propto T_c^{4/3} \), or (Moriya and Kawabata, 1973)
\[ T_c \propto (-t)^{3/4}. \tag{3.51} \]

This result is due to the dangerous irrelevancy of \( u \); in its absence one would have \( T_c \propto (-t)^{3/2} \). This result was confirmed by Millis (1993), who showed how to obtain it from an RG analysis of Hertz’s action. The importance of the DIV was stressed by him and also by Sachdev (1997).

These results also determine the behavior of the correlation length \( \xi \propto 1/\sqrt{|\delta|} \). At \( T = 0 \) we have \( \xi(t = 0, T = 0) \propto |t|^{-1/2} \), in agreement with the value of \( \nu \) in Eq. (3.47), whereas at \( t = 0 \) we have \( \xi(t = 0, T = 0) \propto T^{-\gamma_T} \) with (Millis, 1993)
\[ \nu_T = 2/3. \tag{3.52} \]

For general \( d, \nu_T = (d + 1)/6 \).

(ii) Magnetic susceptibility: Now consider the magnetization susceptibility \( \chi \). The scaling law is
\[ \chi(t, T) = b^{7/4} F_\chi(tb^{1/4}, Tb^2, ub^{-(d-1)}) \]
\[ = b^{7/2} F_\chi(tb^2, Tb^3, ub^{-(d-1)}). \tag{3.53} \]

Since we are dealing with a Gaussian theory \( \delta(t, T) \sim 1/\sqrt{\chi(t, T)} \), and we know from the behavior of \( \delta(t, T) \) that \( u \) is dangerously irrelevant, viz.
\[ F_\chi(0, 1, y \to 0) \propto 1/y. \tag{3.54} \]
At \( t = 0 \) in \( d = 3 \) we thus have \( \chi(t = 0, T = 0) \propto T^{-\gamma_T} \) with
\[ \gamma_T = 4/3, \tag{3.55} \]
and more generally \( \gamma_T = (d + 1)/3 \). At \( T = 0 \), on the other hand, we have \( F_\chi(1, 0, y \to 0) \propto \text{const} \), and hence \( \chi(t = 0, T = 0) \propto |t|^{-1} \) in agreement with the value of \( \gamma \) in Eq. (3.47).
(iii) Magnetization: The magnetization $m$ obeys a scaling law

$$m(t, T) = b^{-(d+z-2+\eta)/2} F_m(t^{1/\nu}, T^{\beta}, u^{b-(d-1)}) = b^{-(d+1)/2} F_m(t^2, T^3, u^{b-(d-1)}).$$  (3.56)

In general, $m$ is affected by the DIV $u$, just as $\gamma$ is. However, at $t = 0$ this is not the case: Since $m \propto \sqrt{d}/u$, and $\delta(t = 0) \propto u$, see Eq. (3.49b), $u$ drops out and hyperscaling works. We thus have

$$m(t = 0, T) = b^{-(d+1)/2} F_m(0, T^3, 0),$$  (3.57)

or $m(t = 0, T) \propto T^{2/3}$ with $\beta_T = (d + 1)/6$ in general, or

$$\beta_T = 2/3$$  (3.58)

for $d = 3$. In interpreting this exponent one needs to keep in mind that the magnetization is nonzero only for $-t > u T^{4/3}$ (putting a constant equal to unity); see Eq. (3.50). For $-t \gg u T^{4/3}$ one observes static scaling with small-temperature corrections, and for $-t < u T^{4/3}$ the scaling function vanishes identically. The exponent $\beta_T$ therefore cannot be observed via the $T$ dependence of $m$ at $r = 0$. It does, however, determine the more general scaling form of $m$ as a function of $t$ and $T$; see Kirkpatrick and Belitz (2015a).

We also note that combining $m \propto \sqrt{d}/\delta$ with $\delta \propto T^{4/3} - T_c^{4/3}$, which follows from Eq. (3.49b), yields (Moriya, 1985)

$$m^2 \propto T_c^{4/3} - T^{4/3}. $$  (3.59)

(iv) Specific heat: The free-energy density $f$ obeys a homogeneity law

$$f(t, T) = b^{-(d+z)} F_f(t^{1/\nu}, T^{\beta}).$$  (3.60)

For the specific-heat coefficient $\gamma = -\partial^2 f / \partial T^2$ this implies

$$\gamma(t, T) = b^{3-d} F_f(t^{2}, T^{3}).$$  (3.61)

For $d = 3$ scaling thus yields $\gamma = \text{const}$. An explicit calculation of the Gaussian fluctuation contribution to the free energy (Brinkman and Engelsberg, 1968; Makoshi and Moriya, 1975; Millis, 1993; Lonzarich, 1997) yields

$$\gamma(t, T) = \ln t, \quad \gamma(t = 0, T) \propto \ln T.$$  (3.62)

For general $d > 1$, the exponents $\bar{\alpha}$ and $\bar{\alpha}_T$ (for a definition, see Appendix B) are $\bar{\alpha} = (3 - d)/2$ and $\bar{\alpha}_T = (3 - d)/3$.  

(v) Electrical resistivity: In order to discuss relaxation rates, we start with the single-particle rate $1/\tau_{sp}$. This is dimensionally an energy, and hence has a scale dimension $[1/\tau_{sp}] = z = 3$. The relevant homogeneity law is

$$1/\tau_{sp}(t, T) = b^{-z} F_{\tau}(t^{1/\nu}, T^{\beta}) = T F_{\tau}(1, T/t^{1/2}).$$  (3.63)

---

28These exponents describe the leading fluctuation contribution to the specific-heat coefficient. For $d > 3$ the latter is subleading compared to a constant nonscaling contribution.

At $t = 0$ we have $1/\tau_{sp}(t = 0, T \rightarrow 0) \propto T$. For $t \neq 0$ we must recover the Fermi-liquid result $1/\tau_{sp} \propto T^2$, which implies $F_{\tau}(1, x = 0) \propto x$, or $1/\tau_{sp} \propto T^2/t^{1/2}$.

The transport rate $1/\tau_T$, which determines the electrical resistivity, is also dimensionally an energy, but its scale dimension is not equal to $z$. The reason is the backscattering factor in the Boltzmann equation, which provides an extra factor of $k^2 \sim b^{-2}$, with $k$ the hydrodynamic wave number.  

We thus have $[1/\tau_T] = z + 2$, and the homogeneity law for the resistivity $\rho$ is

$$\rho(t, T) = b^{-(z+2)} F_{\rho}(t^{1/\nu}, T^{\beta}) = T^{5/3} F_{\rho}(1, T/t^{3/2}).$$  (3.64)

At $t = 0$ we recover the result of Mathon (1968): $\rho(t = 0, T \rightarrow 0) \propto T^{5/3}$. For $t \neq 0$ we can again invoke the Fermi-liquid $T^2$ behavior to conclude $\rho \propto T^2/t^{1/2}$.

Note that $\rho$ does not obey naive scaling. While this is true for many of the observables discussed, in this case the reason is not a DIV. Rather, the underlying relaxation rate does not have its naive scale dimension. As in the case of a DIV, this must be established by explicit calculations; it cannot be deduced from general scaling arguments. In the disordered case, this particular complication does not occur; see Sec. III.C.3.e, point (iii).

To summarize, the critical exponents at Hertz’s fixed point, which determine the preasymptotic behavior in a clean system before the first-order nature of the transition becomes manifest, are given for all $d > 1$ by

$$z = 3, \quad \nu = 1/2, \quad \nu_T = (d + 1)/6, \quad \beta = 1/2, \quad \beta_T = (d + 1)/6, \quad \delta = 3, \quad \gamma = 1, \quad \gamma_T = (d + 1)/3, \quad \eta = 0, \quad \bar{\alpha} = (3 - d)/2, \quad \bar{\alpha}_T = (3 - d)/3.$$  (3.65)

---

29This is true for scattering by long-wavelength magnetic excitations. In a Fermi liquid, both $1/\tau_{sp}$ and $1/\tau_T$ scale as $T^2$.

30See footnote 29 for the interpretation of $\bar{\alpha}$ and $\bar{\alpha}_T$ for $d > 3$.  

c. First- and second-order transitions, tricritical behavior, and quantum-critical points

OP fluctuations affect the second-order transition above the tricritical temperature (the line of second-order transitions about TCP in Fig. 37), where the critical behavior is in the appropriate classical universality class: Heisenberg, $XY$, or Ising, depending on the nature of the magnet. Along the wing-critical lines (between TCP and QCP in Fig. 37) the critical behavior is always in the classical Ising universality class, since the presence of a magnetic field effectively makes the OP one dimensional. The tricritical behavior is described by the mean-field theory with logarithmic corrections, as the upper critical dimension for a classical tricritical point is $d_c^z = 3$ (Wegner and Riedel, 1973).
\textit{d. Quantum-critical points at the wing tips}

A magnetic field restores the QCP that is suppressed in zero field: The tricritical wings end in a pair of QCPs (see Fig. 37) in the \( T = 0 \) plane at a point \( (t_c, h_c) \), with \( t_c \) and \( h_c \) the critical values of the control parameter and the magnetic field, respectively. The magnetization is nonzero at this point and has a value \( m_c \). This QCP has the remarkable property that the quantum-critical behavior can be determined exactly. The reason is that the nonzero field and magnetization give the fermionic Goldstone modes a mass, and the field conjugate to the OP therefore does not change the soft-mode structure of the system. Under these conditions, Hertz theory is expected to be valid (Belitz, Kirkpatrick, and Vojta, 2002). In the present case, an expansion in powers of \( \delta m = m - m_c \) about the QCP shows that the quantity \( \delta = 2m, \delta t = \delta h, \) with \( \delta t = t - t_c \) and \( \delta h = h - h_c \) plays the role of the conjugate field. Switching on an external magnetic field from \( h = 0 \) gives certain soft modes a mass, but changing \( h \) from \( h_c \neq 0 \) does not lead to further changes in the soft-mode spectrum, and neither does changing the value of \( t \). Hertz theory thus gives the exact static quantum-critical behavior, i.e.,

\[
\beta = \nu = 1/2, \quad \gamma = 1, \quad \eta = 0, \quad \delta = 3. \tag{3.66}
\]

The dynamical behavior can be determined as follows. The magnetization at criticality as a function of the conjugate field obeys the homogeneity law (which has the effects of the DIV \( u \) built in)

\[
\delta m(\delta) = b^{\beta/\nu} F_{\delta m}(\delta h^{\delta/\nu}). \tag{3.67}
\]

With mean-field values for the exponents this yields \( \delta m \propto \delta^{1/3} \). But \( \delta t \), and therefore \( \delta \), within Hertz theory scales as \( T^{(d+1)/3} \); see Eq. (3.49b) and its generalization to a general \( d \). We thus find that, at the QCP as a function of \( T \), the magnetization decreases as

\[
\delta m(T) = m(T) - m(T = 0) \propto -\delta h^{1/3} \propto -T^{(d+1)/9}, \tag{3.68}
\]
or \( T^{d/9} \) in \( d = 3 \). This is the result obtained by Belitz, Kirkpatrick, and Rollbühler (2005) using different arguments. This reasoning has the advantage that it also immediately yields the behavior of the magnetic susceptibility \( \chi \), which is easier to measure. It obeys

\[
\chi(\delta) = b^{\gamma/\nu} F_{\chi}(\delta h^{\delta/\nu}), \tag{3.69}
\]

which yields for the \( T \) dependence of \( \chi \) at the QCP

\[
\chi(T) \propto T^{-(d+1)/9}, \tag{3.70}
\]
or \( T^{-8/9} \) in \( d = 3 \). We stress again that this is the exact quantum-critical behavior.

We also mention that in the presence of weak quenched disorder, weak enough for the tricritical wings to still be present (see Fig. 37 and the related discussion), the asymptotic critical behavior is unknown. In a transient preasymptotic region the behavior is governed by Hertz’s fixed point for disordered systems; for a discussion of preasymptotic behavior, see Kirkpatrick and Belitz (2015a). However, this fixed point is ultimately unstable since it violates the Harris criterion and the true critical fixed point may be of a strong-disorder type.

\textit{e. Comparison with experiment}

In order to compare these theoretical predictions with experiments, we recall that the theory states that if there is a QPT to a homogeneous FM state in a clean bulk system, then the transition is first order. This qualification is important for several reasons: (1) The transition at low temperatures may be to a different state; see Secs. II.D, II.E, and III.E. (2) The presence of quenched disorder has a qualitative effect on the transition, and sufficiently strong disorder will always render the transition second order; see Secs. III.B.3 and III.C.2. (3) The fermionic soft modes that drive the transition first order exist only in 2D and 3D systems; the theory therefore does not apply to quasi-1D materials.

With these caveats taken into account, we consider the systems listed in Tables I and II. With one exception, these are all rather clean systems that show a first-order transition, as expected. The only questionable case is YbCu\(_2\)Si\(_2\), which is strongly disordered; however, the nature of the magnetic order is not clear. In the rather clean system URhAl a tricritical point is suspected but so far has not been conclusively observed (Combier, 2013).

The materials in Table III comprise systems that are rather clean, with residual resistivities comparable to those in Tables I and II, yet show a second-order transition. The behavior observed in these systems is consistent with the preasymptotic critical behavior governed by Hertz’s fixed point discussed in Sec. III.C.2.b. In particular, the characteristic \((-t)^{3/4}\) behavior of the Curie temperature, Eq. (3.51), was observed as early as 1975 by Sato (1975) in \( (\text{Ni}_{1-x}\text{Pd}_x)_3\text{Al} \), and the behavior of the specific-heat coefficient is consistent with Eq. (3.62). For a more recent observation of the scaling of \( T_C \), see Fig. 14. The most obvious interpretation of these observations is that these experiments indeed probe the preasymptotic region, and following \( T_c \) to lower values would reveal a tricritical point and a first-order transition at the lowest temperatures. This expectation is supported by the fact that the lowest \( T_c \) observed so far in these systems is relatively high, and by the observation that \( T_c \) at intermediate temperatures also follows the \((-t)^{3/4}\) law in systems where the transition at asymptotically low temperatures is known to be first order, for instance, in MnSi (Pfleiderer et al., 1997). An experimental confirmation or otherwise of this expectation would be important.

Another experimental check of the theory would be the critical behavior at the wing tips, Eqs. (3.66), (3.68), and (3.70), which has not been studied so far.

\textit{3. Disordered systems}

For disordered systems, the situation is qualitatively different since the transition is continuous at the mean-field level. While the development of the effective action proceeds in exact analogy to Sec. III.C.2, the final result is a stable critical fixed point where the asymptotic critical behavior is not given
by power laws due to the existence of marginal operators (Belitz et al., 2001a, 2001b).

a. Effective soft-mode action

In a disordered system, the relevant fermionic soft modes are the diffusons discussed in Sec. III.A.2. Their effective action can be written in a closed form, namely, the matrix nonlinear sigma model developed by Finkelstein (1983) for studying the Anderson-Mott metal-insulator transition problem [for reviews, see Belitz and Kirkpatrick (1994) and Finkelstein (2010)]. The quenched disorder is handled technically by means of the replica trick (Edwards and Anderson, 1975; Grinstein, 1985). If one describes the soft modes by \( q^{\alpha \beta}(q) = \Theta(-nm) \sum_{m} q^{\alpha \beta}_{nm}(k; q) \), with \( Q_{nm} \) from Eqs. (3.16) and \( \alpha, \beta \) replica indices, it can be written

\[
A_{\text{diffs}}[q] = A_{\text{NLMS}}[q] = -\frac{1}{2G} \int d\Omega \langle \hat{\nabla} \hat{Q}(x) \rangle^2 + 2H \int d\Omega \hat{Q}(x)
\]

\[
+ A_{\text{int}}[(\pi N_{\text{F}} \hat{Q})/2] + A_{\text{int}}[(\pi N_{\text{F}} \hat{Q})/2],
\]

where

\[
\hat{Q} = \left( \sqrt{1-q^2}, \frac{q}{q^2}, -\sqrt{1-q^2} \right)
\]

is a nonlinear function of \( q \), and \( \Omega \) is a frequency matrix with elements

\[
\Omega_{\alpha \beta} = (\pi_0 \otimes s_0)\delta_{\alpha \beta} \omega_{n_\alpha}.
\]

Here \( 1 \equiv (n_1, m_1), \) etc., and \( \text{tr} \) denotes a trace over all discrete degrees of freedom (frequency, spin, particle-hole, and replica). The coupling constant \( G \) is proportional to the bare (i.e., Boltzmann) resistivity, and thus is a measure of the disorder strength. \( H \) is proportional to the specific-heat coefficient. The first two terms in Eq. (3.71a) describe noninteracting electrons. They are the fermionic version (Efetov, Larkin, and Khmelnitskii, 1980) of Wegner’s nonlinear sigma model for the Anderson localization problem (Wegner, 1979). Note the diffusive structure of these two terms once they are expanded to \( O(q^2) \), with the gradient-squared scaling as a frequency. The last two terms reflect the electron-electron interactions in the spin-singlet and spin-triplet channels, respectively. They are quadratic in \( \hat{Q} \) with coupling constants \( K_s \) and \( K_t \), respectively, and are effectively linear in the frequency (Finkelstein, 1983). They therefore do not spoil the soft-mode structure of the nonlinear sigma model but just renormalize the prefactor of the frequency in the diffusion pole. The magnetization again couples linearly to the electron spin density, the soft part of which is linear in \( \hat{Q} \). The coupling term reads

\[
A_{\text{c}} = \frac{\sqrt{\pi T c}}{m} \int d\Omega \sum_{\alpha \beta} \sum_{n} \sum_{k} m^{\alpha \beta}_n(x) \sum_{r=0,3} (-1)^r \nabla_{m+n}^{\alpha \beta}(x)
\]

\[
\times \sum_{m} \text{tr}[(\tau_r \otimes s_1)\hat{Q}_{m,m+n}^{\alpha \beta}(x)],
\]

with \( c \) a coupling constant. \( m^i (i = 1, 2, 3) \) denotes again the three components of the order-parameter field \( m \), which now also carries a replica index \( \alpha \). It determines the physical magnetization \( m \) via

\[
m = \mu_B \sqrt{\pi N_{\text{F}}^3 T / K_t(m^{\alpha \beta}_{n=0}(x))}.
\]

The order-parameter action is similar to the one in the clean case, Eq. (3.43), but there is an additional quartic term that arises from the quenched disorder,

\[
A_{\text{OP}} = -\sum_{k,n} m^{\alpha \beta}_n(k) [t + a k^2 + b(\Omega_n)^2] \cdot m^{\alpha \beta}_{-n}(-k)
\]

\[
+ u \int d\Omega \sum_{n_1,n_2,n_3} \sum_{\alpha} \left[ m^{\alpha \beta}_{n_1}(x) \cdot m^{\alpha \beta}_{n_2}(x) \right]
\]

\[
\times \left[ m^{\alpha \beta}_{n_3}(x) \cdot m^{\alpha \beta}_{-n_1,-n_2,-n_3}(x) \right]
\]

\[
+ v \int d\Omega \sum_{n_1,n_2,n_3} \sum_{\alpha} \left[ m^{\alpha \beta}_{n_1}(x) \right]^2 \left[ m^{\alpha \beta}_{n_2}(x) \right]^2.
\]

The last term, with coupling constant \( v \), is a random-mass or random-temperature term that arises from the disorder dependence of the bare distance from criticality whose disorder average is given by \( t \). There also is a term cubic in \( m \), which carries at least one gradient or frequency and is less relevant for the critical behavior than the terms shown.

The soft-mode action given by Eqs. (3.71)–(3.74) was motivated and derived by Belitz et al. (2001a) from an underlying microscopic fermionic action. However, such a derivation is not necessary. All parts of the effective action can be obtained from more general considerations, namely, (1) the existence of an effective soft-mode theory for disordered interacting electrons, (2) symmetry considerations for a quantum \( \phi^4 \) theory with a vector OP, and (3) the Zeeman coupling between the OP and the electron spin density. In particular, the OP part of the action can be either written down based on symmetry considerations or derived by means of a Hubbard-Stratonovich decoupling of the particle-hole spin-triplet interaction term in the underlying fermionic action. In the latter case, a spin-triplet interaction will be generated again by renormalization in the fermionic sector as long as a spin-singlet interaction is present. The presence of the last term in Eq. (3.71a) therefore does not constitute any double counting.

b. Hertz’s action

As in the clean case, if we keep only the term of \( O(Mq) \) in Eq. (3.72) and integrate out the fermions, we recover Hertz’s action (Hertz, 1976) (plus the random-mass term, which was not considered by Hertz). The Landau-damping term now has the form \( \Omega / |k|^2 \) due to the diffusive nature of the fermionic soft modes. The paramagnon propagator thus reads

\(\text{Footnote 31:}\) The term \( b(\Omega_n)^2 \) in Eq. (3.74) was erroneously written as \( b|\Omega_n| \) in Belitz et al. (2001a). This is of no consequence, as the term in question is RG irrelevant in either case.
\[
\langle M^u_n(k)M^v_n(p) \rangle
\]
\[
= \delta_{k-p} \delta_{n-m} \delta^{d_{ij}} \delta^{dp} \frac{1}{2 t + a k^2 + b(\Omega_n)^2 + Gc|\Omega_n|/(k^2 + GH|\Omega_n|)}.
\]

(3.75)

Dropping the random-mass terms, the action becomes

\[
A_{\text{Hertz}} = -\sum_{k,n} \sum_a m^u_n(k) [t + a k^2 + Gc|\Omega_n|/|k|^2] \times m^v_n(-k) \\
+ u_4 \int dx T \sum_{n_1,n_2,n_3} \sum_a \left[ m^u_n(x) \cdot m^v_n(x) \right] \times \left[ m^u_{n_1}(x) \cdot m^v_{n_1-n_2-n_3}(x) \right].
\]

(3.76)

Power counting again suggests a continuous phase transition with mean-field static critical exponents, only now the upper critical dimension is \(d_u^c = 0\), and the dynamical critical exponent is \(z = 4\). This fixed point is ultimately unstable, since the same physics that leads to the Landau-damping term also leads to the terms of higher order in \(q\) in Eq. (3.72).

Nevertheless, as in the clean case (see Sec. III.C.2) it is important to study this fixed point since it is experimentally relevant in a preasymptotic crossover region (Kirkpatrick and Belitz, 2014). In the disordered case this is true \textit{a fortiori} since the effects that destabilize Hertz’s fixed point still result in a continuous transition, albeit with different exponents.

The homogeneity relations and exponents for Hertz’s action are obtained by a straightforward modification of the development in Sec. III.C.2.b. The dynamical critical exponent is now

\[ z = 4, \quad (3.77a) \]

which yields an upper critical dimensionality \(d_u^c = 0\), and the DIV \(u\) has a scale dimension \([u] = -d\). For all \(d > 0\) the exponents are [cf. Eqs. (3.65) for the clean case]

- \(\nu = 1/2\), \(\nu_T = (d + 2)/8\),
- \(\beta = 1/2\), \(\beta_T = (d + 2)/8\),
- \(\delta = 3\),
- \(\gamma = 1\), \(\gamma_T = (d + 2)/4\),
- \(\eta = 0\),
- \(\bar{\alpha} = (4-d)/2\), \(\bar{\alpha}_T = (4-d)/4\). \(\quad (3.77b)\)

For \(d > 4\), \(\bar{\alpha}\) and \(\bar{\alpha}_T\) describe the leading fluctuation contribution to the specific-heat coefficient; see footnote 28 for the analogous statement in the clean case.

As mentioned, these exponents do not describe the physical asymptotic critical behavior. Another indication of this is the value of the correlation-length exponent \(\nu = 1/2\), which violates the requirement \(\nu \geq 2/d\), Eq. (3.36), for all \(d < 4\).

For finding the true asymptotic critical behavior it is preferable to integrate out the fermions, but rather deal with the coupled soft-mode field theory for analyzing the fixed-point structure.

c. Fixed-point action

The lowest-order term that was neglected in Eq. (3.76) is the term of \(O(mq^5)\) in Eq. (3.72). It is easy to see that this generates a renormalization of the 2-point \(m\) vertex that is proportional to \(|k|^{d-2}\). For dimensions \(d < 4\), the gradient-squared term in Eq. (3.74) is therefore not the leading wave-number dependence, and it is convenient to add the generated term to the bare action. In a schematic form that suppresses everything not necessary for power counting, the effective action then reads (Belitz et al., 2001a)

\[
A_{\text{eff}}[m,q] = -\int dx m[t + a_d^{-2} \partial^2_k - a_2 \partial^2_k m + O(\partial^4_k m^2, m^4)] \\
- \frac{1}{G_4} \int dx(\partial_s q)^2 + H \int dx(\partial_s q)^2 + (K_s + K_t)T \int dxq^2 \\
- \frac{1}{G_4} \int dx \partial_s^2 q^4 + H_4 \int dxq^4 + O(Tq^3, \partial^2_s q^6, \Omega q^6) \\
+ \sqrt{Tc_1} \int dxmq + \sqrt{Tc_2} \int dxmq^2 + O(\sqrt{Tmq^4}).
\]

(3.78)

Here the fields are understood to be functions of position and frequency, and only quantities that carry a scale dimension are shown. The bare values of \(G_4\) and \(H_4\) are proportional to those of \(G\) and \(H\), \(K_s\), and \(K_t\) are the coupling constants of the terms of \(O(q^5)\) in the interacting parts of \(A_{\text{NL}}\). A term of order \(Tq^3\) that arises from the same part of the action is not important for the problem of magnetic criticality. It therefore is not shown although its coupling constant squared has the same scale dimension as \(1/G_4\) and \(H_4\). \(c_1\) and \(c_2\) are the coupling constants of the terms that result from expanding \(A_c\) in powers of \(q\). Their bare values are proportional to \(c\).

d. Fixed points, and their stability

The action shown schematically in Eq. (3.78) can be analyzed for critical fixed points by means of standard RG techniques (Ma, 1976). We assign a scale dimension \([L] = -1\) to a length \(L\), and \([\tau] = -z\) to the imaginary time \(\tau\) (with \(z\) to be determined). Under renormalization with a length rescaling factor \(b\), all scaling quantities \(A\) transform according to \(A \rightarrow Ab^{|A}|\). In particular, temperature \(T\) and frequency \(\Omega\) have scale dimensions \([T] = [\Omega] = z\).

It is illustrative to again look for a fixed point that describes the mean-field critical behavior of Hertz theory. To this end, let us look for a fixed point where the coupling constants \(a_{d-2}\) and \(c_1\) are marginal. This results in standard mean-field static critical behavior, and a dynamical exponent \(z = 4\), all of which is consistent with the action given in Eq. (3.76) and with the paramagnon propagator, Eq. (3.75). The requirement that the action be dimensionless leads to \([m] = (d-2)/2\), which makes \(r\) relevant with \([r] = 2\). The critical exponents \(\eta\) and \(\nu\) are thus \(\eta = 0\), and \(\nu = 1/2\). This fixed point is unstable for \(d < 4\), since \(|a_{d-2}| = 4-d\), and \(a_{d-2}\) is thus relevant for all \(d < 4\). This is obvious if one adds the term with coupling constant \(a_{d-2}\) to the bare action, as we have done above, but less so if one chooses the bare value of \(a_{d-2}\) to be zero and
have the physics related to $a_{d-2}$ be generated by the term with coupling constant $c_2$. In that case, a careful analysis of the time scales involved leads to the same conclusion (Belitz et al., 2001a). All of this is consistent with the fact that the mean-field value $\nu = 1/2$ violates the Harris criterion discussed in Sec. III.B.3, see Eq. (3.27), and therefore cannot represent the correct critical behavior in a disordered system.

The previous discussion suggests that one should look for a fixed point where only $c_1$ is required to be marginal, which implies $|m| = 1 + (d - z)/2$. The diffusons will be unaffected by the magnetic transition, and hence the scale dimension of the soft fermion field $q$ is $\eta = (d - 2)/2$. This also implies that there is a diffusive time scale characterized by a dynamical exponent

$$z_{\text{diff}} = 2$$

(3.79)

in addition to the critical dynamical time scale whose exponent we denote by $z_c$. This presence of more than one time scale complicates the power-counting arguments, as the scale dimensions of various coupling constants can depend on the context they appear in. That is, the scale dimension $z$ of the various factors of temperature or frequency in the effective action can be equal to $z_{\text{diff}}$ or $z_c$, depending on the context. In particular, $a_{d-2}$ can be irrelevant if the paramagnon propagator appears as an internal propagator in the loop expansion, while it will be marginal in the critical paramagnon propagator, which implies $|m| = 1$. This leads to $z_c = d$ and $\eta = 4 - d$. This makes $a_2$ irrelevant, while $t$ is relevant with $|t| = d - 2$.

The three independent critical exponents thus are

$$\nu = \frac{1}{d - 2}, \quad \eta = 4 - d, \quad z_c = d.$$  

(3.80a)

For this fixed point, $\nu$ satisfies the Harris criterion. The remaining static exponents are given by $d$-dependent generalizations of Eq. (3.25) (Belitz et al., 2001b):

$$\beta = \frac{2}{d - 2}, \quad \gamma = 1, \quad \delta = d/2.$$  

(3.80b)

Equation (3.80b) is valid for $2 < d < 6$. For $d \geq 6$, $\beta$ and $\delta$ lock into their mean-field values; for $\nu$ and $\eta$ this happens for $d \geq 4$. The $T$ dependence of the observables at criticality $t = 0$ is determined by (Belitz et al., 2001b)

$$\beta_T = \beta/2\nu, \quad \gamma_T = \gamma/2\nu.$$  

(3.80c)

To discuss the stability of this fixed point we now consider the remaining coupling constants in the effective action, Eq. (3.78). $c_2$ has a scale dimension $\eta_{c_2} = 1 - z/2$, and thus is irrelevant if $z = z_c$, but marginal if $z = z_{\text{diff}}$. Moreover, due to the existence of two different time scales even some operators that are irrelevant by power counting may effectively act as marginal operators (Belitz et al., 2001b). The reason is that naive power counting is based on a length scale argument, which can be modified if the scale factor $b$ represents a frequency rather than a length. Since the difference between the two dynamical exponents $z_{\text{diff}}$ and $z_c$ is equal to $d - 2$, this implies that coupling constants with a naive scale dimension given by $-(d - 2)$ can act as marginal operators under certain conditions. As a consequence all terms that are shown explicitly in Eq. (3.78) are important for determining the leading critical behavior and constitute the fixed-point action.

### e. Asymptotic critical behavior

The conclusion so far is that the fixed-point action represented by Eq. (3.78) contains marginal operators that result, order by order in a loop expansion, in logarithmic corrections to the fixed point with critical exponents given by Eqs. (3.80). The remaining question is what is the result if the loop expansion is summed to all orders?

(i) Integral equations for the diffusion coefficients: This question can be answered exactly without resorting to a small parameter (such as an expansion in $\epsilon = d - 4$) (Kirkpatrick and Belitz, 1996; Belitz et al., 2001a, 2001b). This hinges on various properties of the loop expansion: First, at the fixed point of interest the fermionic dynamics remain diffusive. The coupling constants $K_s$ and $K_t$ do not change this and therefore can be ignored. Second, $G$ and $c_2$ are not singularly renormalized. Third, the renormalized versions of $G_A$ and $H_A$ are proportional to those of $G$ and $H$, as are their bare values. Finally, $c_1$ is held fixed by definition of the fixed point. This leaves the renormalizations of $H$ and the two-point order-parameter vertex $u_2 = t + a_{d-2} k^{d-2} + a_k k^2$ to be determined.

The resummation of the loop expansion to all orders can be expressed in terms of two coupled integral equations for $H$ and $u_2$, or, equivalently, for the thermal diffusion coefficient $D(\Omega) = 1/GH(\Omega)$ and the spin diffusion coefficient $D_J(k, \Omega) = 16\pi u_2(k, \Omega)/Gc_t^2$, both of which acquire a frequency dependence under renormalization. $D$ and $D_J$ simultaneously go to zero at a critical value of $G$, and in the vicinity of that critical point the integral equations can be solved analytically. It turns out that the logarithmic corrections obtained in perturbation theory do not change the power laws given in Eqs. (3.80), but rather result in log-normal corrections to power-law scaling. For instance, the magnetization $m(t, T = 0) = d$ has an asymptotic behavior (Belitz et al., 2001b)

$$m(t \to 0) \propto |t|/|t|^{|\beta|},$$  

(3.81a)

with $\beta = 2$ from Eqs. (3.80) and

$$g(x \to \infty) \propto e^{(\ln x^2)^{1/2}/\ln(3/2)}.$$  

(3.81b)

Similarly, at $t = 0$ as a function of a magnetic field $h$,

$$m(t = 0, h \to 0) \propto |h|^{1/3},$$  

(3.81c)

with $\delta = 3/2$ from Eqs. (3.80). The specific heat also has a log-normal critical behavior. However, the critical exponent $\gamma$ comes without logarithmic corrections; the magnetic susceptibility diverges as

$$\chi(t \to 0) \propto 1/|t|.$$  

(3.81d)

(ii) Scaling considerations: Thermodynamic quantities. All of these results are conveniently summarized in the following
generalized homogeneity law for the free-energy density (Belitz et al., 2001b):

\[ f(t, T, h) = b^{-(d + z_g)} f_1(t b^{1/\nu}, T b^{\gamma_g}, h b^{\delta_g}) + b^{-(d + z_g)} f_2(t b^{1/\nu}, T b^{\gamma_g}, h b^{\delta_g}). \]  \hfill (3.82)

Here \( z_g \) is the dynamical exponent, which determines \( f_1(t, T, h) \) the temperature dependence of the specific heat, and \( z_g \) is the dynamical exponent due to the generic soft modes, which determines the temperature dependence of the order parameter and its susceptibility. If \( z_c \geq z_g \) (this has to be the case, see Sec. III.C.4), we obtain homogeneity laws for the order parameter with the temperature dependence of the specific heat, and \( f_1(t, T, h) \) with \( z_g \):

\[ \lambda = \ln g(\ln b) / \ln b, \]  \hfill (3.86)

with \( g(\ln b) \) from Eq. (3.81b).

This critical behavior is expected to be exact provided a continuous transition into a homogeneous FM phase occurs. However, rare-region effects may mask this critical behavior. Theories that deal with such effects are discussed in Secs. III.D and IIE.

(iii) Scaling considerations: Electrical resistivity. We finally mention the electrical resistivity \( \rho \). The transport relaxation rate is dominated by the disorder, which is unaffected by the magnetic ordering. The scale dimension of \( \rho \) at a ferromagnetic QCP is therefore zero. However, \( \rho \) depends on the critical dynamics, since the paramagnon propagator enters the calculation of \( \rho \) in perturbation theory. From Eq. (3.78) we see that one-loop corrections to \( \rho \) can be constructed, for instance, from one vertex with coupling constant \( 1/G_0 \), or from two vertices with coupling constant \( 2c_2 \). These terms belong to the class of least irrelevant variables with respect to the critical fixed point; their scale dimension is \( -(d-2) \). Denoting the least irrelevant variables collectively by \( u \), we have the following homogeneity law for the resistivity:

\[ \rho(t, T) = F_{\rho}(t b^{1/\nu}, T b^{\gamma_g}, u b^{-(d-2)}) = \text{const} + b^{-(d-2)} F_{\rho}(t b^{1/\nu}, T b^{\gamma_g}), \]  \hfill (3.87)

where we used the fact that the leading correction to \( \rho \) is linear in \( u \). At criticality, this yields

\[ \rho(t = 0, T) \propto T^{(d-2)/\nu}. \]  \hfill (3.88)

For the \( t \) dependence at \( T = 0 \) there are additional logarithmic complications due to a resonance between the scale dimensions of \( u \) and \( t \); see Belitz et al. (2001b).

Alternatively, one can argue that \( \rho \) consists of a background contribution that does not scale, and a singular contribution \( \delta \rho \) that does. Since \( \rho \) is dimensionally length to the power \( d - 2 \), one expects

\[ \delta \rho(t, T) = b^{-(d-2)} F_{\rho}(t b^{1/\nu}, T b^{\gamma_g}), \]  \hfill (3.89)

which again yields Eq. (3.88). Note that this argument builds in the DIV \( u \), so naive scaling works.

f. Preasymptotic behavior

The logarithmic nature of the asymptotic critical behavior described previously suggests that it is valid only in an exponentially small region around the critical point. Indeed, a numerical solution of the integral equations mentioned in Sec. III.C.3.e shows that the behavior in an observable region around criticality is given by effective power laws that correspond to the quantity \( \lambda \) defined in Sec. III.C.3.e being \( \lambda \approx 2/3 \) in a large range of scales (Kirkpatrick and Belitz, 2014). For instance, the specific-heat coefficient follows effective power laws with exponents (see footnote 5)

\[ \alpha^{\text{eff}} \approx 0.4, \quad \alpha_T^{\text{eff}} \approx 0.18 \]  \hfill (3.90a)
over almost three decades. Similarly, the critical temperature dependence of the spin susceptibility and the magnetization is given by effective exponents

$$\gamma^\text{eff}_T \approx 0.625, \quad \beta^\text{eff}_T \approx 0.75, \quad (3.90b)$$

and the corresponding effective static exponents are

$$\gamma = 1, \quad \rho^\text{eff} \approx 1.2, \quad \delta^\text{eff} \approx 1.83. \quad (3.90c)$$

For the exponent that determines the shape of the phase diagram in the $T-t$ plane, Eq. (3.83d), we have

$$(\nu_{\text{cg}})^\text{eff} \approx 1.6. \quad (3.90d)$$

Only the value of $\gamma$ is the same in the preasymptotic and asymptotic regions, respectively. This is important for the interpretation of experiments.

g. Summary of critical exponents in the disordered case

In summary, the critical exponents for the disordered case in $2 < d < 4$ dimensions in both the asymptotic and preasymptotic regions are given by Eqs. (3.84) and (3.85). In the asymptotic regime they do not represent pure power-law behavior since $\lambda$ is the scale-dependent object given in Eq. (3.86). In the preasymptotic regime $\lambda \approx 2/3$, and the exponents represent effective power laws.

h. Relation to experiment

The interpretation of experiments on strongly disordered systems is difficult for various reasons. First, the control parameter tends to be the chemical composition, which necessitates the preparation of a separate sample for each data point. This makes the precise determination of the critical point difficult, and neither the precision nor the absolute values of the distance from criticality are anywhere near the values that in classical systems are known to be necessary for a reliable determination of critical exponents. Second, the Griffiths-region effects discussed in Sec. III.D are expected to be pronounced in strongly disordered systems and coexist with critical phenomena.

A well-studied strongly disordered system is URu$_{2-x}$Re$_x$Si$_2$; see Sec. II.C.2. Bauer et al. (2005) found a QCP at $x \approx 0.3$; scaling plots yielded exponent values $\delta = 1.56$ and $\beta_T = 0.9$. $\gamma_T$ was inferred from the Widom relation [which holds in this context, see Kirkpatrick and Belitz (2015a)], $\gamma_T = \beta_T (\delta - 1) = 0.5$. The specific-heat coefficient showed a $\ln T$ behavior over a wide range of $x$ values. A later analysis (Butch and Maple, 2009) put the critical concentration at $x \approx 0.15$ and found continuously varying exponents in the range $0.6 \geq x \geq 0.2$, with $\delta \rightarrow 1$, $\gamma_T \rightarrow 0$, and $\beta_T \approx 0.8$ roughly constant. If the data represent critical phenomena, then continuously varying exponents are hard to understand. Also, an exponent $\gamma_T = 0$, which must signal a divergence of the OP susceptibility that is only logarithmic, would be very unusual.

In Ni$_{1-x}$V$_x$ (Sec. II.E.1.c), Ubaid-Kassis and Schroeder (2008) found a critical point at $x_c \approx 0.11$ with $\gamma_T = 0.37 \pm 0.07$, $\beta_T \approx 0.5$, and $\delta = 1.8 \pm 0.2$. The value of $\delta$ agrees very well with Eqs. (3.90); the agreement for $\gamma_T$ and $\beta_T$ is less satisfactory. These data were reinterpreted by Ubaid-Kassis, Vojta, and Schroeder (2010) in terms of a Griffiths phase for $x < x_c$.

Finally, the exponent that governs the scaling of $T_c$ with the control parameter is equal to 2 asymptotically, and about 1.6 in the preasymptotic region; see Eqs. (3.83d) and (3.90d). This is in contrast to the result from Hertz theory in the clean case, where the corresponding value is 3/4; see Eq. (3.51). An exponent much greater than 1 is qualitatively consistent with the tail in the phase diagram observed in many disordered systems (see Figs. 33 and 35), and also with the schematic phase diagram shown in Fig. 38. As discussed in Sec. II.E, these tails are often interpreted as signalizing quantum Griffiths effects. These two interpretations are not mutually contradictory; more detailed experimental investigations are needed to distinguish between them.

4. Exponent relations

At a classical critical point, only two static critical exponents are independent. This implies that there must exist relations between various exponents. These relations have a complicated history, and some of them were initially found empirically (Stanley, 1971). Several of them take the form of a rigorous inequality that turns into an equality if certain conditions are fulfilled. Well-known examples are Widom’s equality $\gamma = \beta (\delta - 1)$ (Widom, 1964) and Fisher’s equality $\gamma = (2 - \eta) \nu$ (Fisher, 1964). Relations between the exponents at a QCP defined in Appendix B have been derived and discussed by Kirkpatrick and Belitz (2015a).

D. Rare-region effects in disordered systems

1. Quantum Griffiths effects

The notion of a Griffiths phase is well established in both classical and quantum disordered systems (Griffiths, 1969; McCoy, 1969; Randeria, Sethna, and Palmer, 1985; Bray, 1987; Millis, Morr, and Schmalian, 2002; Vojta, 2010). The basic idea can be illustrated by considering a classical randomly diluted Ising FM in $d$ dimensions. In this model some of the FM bonds are missing with a probability $p$. As a result, the critical temperature $T_c$ in the random system is lower than the corresponding critical temperature in the pure or nonrandom system. In random systems, the latter is often denoted by $T_G$ and referred to as the Griffiths temperature. In general, interesting effects occur both in the “paramagnetic” phase $T > T_G$ and in the “Griffiths phase” $T_c < T < T_G$. Here we focus on the latter.

3These exponent relations are also often referred to as “scaling relations” or “scaling laws,” the latter not to be confused with the homogeneity laws that are often referred to by the same term.

The relevant concepts were put forth simultaneously by Griffiths (1969) and McCoy (1969). McCoy considered a strip-random two-dimensional classical model (McCoy and Wu, 1968) that is closely related to the quantum-mechanical problem of a random transverse-field Ising spin chain (Fisher, 1995). This observation and phenomena deriving from it are often referred to as (quantum) Griffiths-phase effects.
Griffiths argued that in such a system there always exist regions of linear size $L$ that happen to contain no missing bonds, and thus behave as a region of the same size in the corresponding pure system. This is true even for arbitrarily large $L$, but the probability of finding a large region devoid of missing bonds is exponentially small,

$$P(L) \propto \exp(-cL^d). \quad (3.91)$$

Here $d$ is the spatial dimensionality of the system, and $c$ is a constant. If the size of these rare regions is large compared to the local correlation length, then it is meaningful to speak of them as being magnetically ordered. At criticality in a mean-field theory, the correlation length $\xi$ as a function of an applied magnetic field $h$ scales as $\xi \sim h^{1/3}$. This in turn suggests that in the entire Griffiths phase there is a contribution $\delta F_G$ to the free energy that reflects both the exponentially small probability of rare regions and the scaling of the correlation length with the magnetic field (Dotsenko, 2006):

$$\delta F_G \propto \exp(-c'h^{-d/3}). \quad (3.92)$$

with $c'$ another constant. That is, in the entire Griffiths phase the free energy is a nonanalytic function of the field $h$ at $h = 0$. However, the singularity is only a very weak essential one.

The weak singularities in the thermodynamic properties in the classical Griffiths phase are difficult to detect experimentally. However, the existence of ordered rare regions has a qualitative effect on the dynamics of the equilibrium time-correlation functions. This is physically obvious since overturning large clusters of ordered spins takes a time that grows exponentially with the size of the cluster, and time-correlation functions in the Griffiths phase will depend on such dynamical processes. We have given the qualitative argument in Sec. III.A.1. The result, Eq. (3.7b), was that time-correlation functions are expected to decay slower than any exponential.

The conclusion is that the static effects in the classical Griffiths phase are very weak, but dynamic Griffiths effects are quite profound, changing exponential decay of time-correlation functions into nonexponential decay. As stressed in Sec. I, in quantum statistical mechanics the statics and the dynamics are coupled. This implies that Griffith-phase effects are expected to be important for both the statics and the dynamics near QPTs in disordered systems in general, and in disordered quantum FMs, in particular. In fact, it turns out that in the quantum case the dynamical singularities are even stronger than suggested by the classical arguments.

In the context of quantum mechanics, this goes back to the model proposed and studied by McCoy and Wu (1968) (McCoy, 1969; McCoy and Wu, 1969), which is closely related to a 1D quantum problem. This model was later generalized (McCoy, 1970; Shankar and Murthy, 1987), and its quantum-mechanical interpretation was studied in detail by Fisher (1992, 1995) and others (Rieger and Young, 1996; Young, 1997; Pich et al., 1998). The crucial point is that the slow dynamics associated with the Griffiths phase greatly affects the zero-temperature behavior. To see this, consider a local magnetized rare region of linear size $L$, separated by a domain wall from the rest of the system as in the classical case. Its imaginary-time local dynamic susceptibility will decay exponentially by a quantum tunneling process. For long imaginary times we have

$$\chi_{\text{loc}}(\tau \to \infty) \propto \exp[-\tau/\bar{t}(L)]. \quad (3.93)$$

where $\bar{t}(L)$ is the characteristic relaxation time for the tunneling process. To estimate $\bar{t}(L)$ we imagine a domain wall in imaginary-time space for a cluster of size $L^d$ in real space. This was considered for Ising systems (Thill and Huse, 1995; Guo, Bhatt, and Huse, 1996; Rieger and Young, 1996; Pich et al., 1998; Motrunich et al., 2000; Millis, Morr, and Schmalian, 2002) and for Heisenberg magnets (Vojta and Schmalian, 2005); for a review, see Vojta (2010). Most of the work on this topic was done for AFMs, i.e., the case of a nonconserved OP. One finds

$$\bar{t}(L) \sim \tau_0 \exp(\bar{\sigma}L^d). \quad (3.94)$$

Here $\tau_0$ is a microscopic time scale, $\bar{\sigma}$ is a constant, and the overbars distinguish $\bar{t}$ and $\bar{\sigma}$ from the corresponding quantities in the classical case, Eq. (3.6). Effectively, in the quantum case the volume of the region is $L^{d+z-}$, with $z$ the dynamical exponent, and the domain wall is a hypersurface with area $L^{d+z-} = L^d$. Physically, the decay of the rare region in the quantum case is much slower than its classical counterpart, Eq. (3.6), since at $T > 0$ the cluster can flip via thermal activation in addition to quantum tunneling. Equations (3.91), (3.93), and (3.94) imply for the average local dynamic susceptibility

$$\chi_{\text{loc}}^{\text{av}}(\tau) = \int_0^\infty dL \exp[-cL^d - (\tau/\tau_0)e^{-\bar{\sigma}L^d}]. \quad (3.95a)$$

In this case the typical length scale is $L_{\text{typ}} \propto [\ln(\tau/\tau_0)]^{1/d}$, and in the limit of large imaginary time the method of steepest descent yields

$$\chi_{\text{loc}}^{\text{av}}(\tau \to \infty) \propto (\tau/\tau_0)^{-c/\bar{\sigma}}. \quad (3.95b)$$

We see that quantum mechanics leads to a power-law decay. This is in contrast to the classical case; see Sec. III.A.1. The $T$ dependence of the static susceptibility is

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34Griffiths effects also exist in the ordered phase. However, they are weaker than the corresponding effects in the disordered phase except in certain special models (Sentil and Sachdev, 1996; Motrunich et al., 2000). Here we focus on the disordered phase.

35Strictly speaking the considerations presented here are valid only for non-Ising metallic magnets, i.e., systems with an order-parameter dimensionality $n > 1$. The reason is that the $|\Omega_n|$ term in the Gaussian order-parameter action corresponds, at $T = 0$, to a long-ranged $1/\tau^2$ decay in imaginary-time space. Because a one-dimensional Ising model ($n = 1$) can have a phase transition with such an interaction, this implies that there might be a freezing phase transition in imaginary-time space that is not included in the simple Griffiths arguments given here. One-dimensional models with $n > 1$ do not have such a phase transition because they are below their lower critical dimension even with this long-ranged interaction.
\[ \chi_{\text{loc}}^n(T \to 0) = \int_0^{1/T} d\tau \chi_{\text{loc}}^n(\tau) \sim T^{c/\beta - 1}. \]  

(3.96)

The conclusion is that Griffiths-phase dynamical singularities lead to low-\( T \) singularities in static quantities. Similarly, the contribution to the specific heat is \( C_{\text{loc}}^n(T) \sim T^{c/\beta} \). If \( c/\beta < 1 \), then these local rare-region contributions dominate the usual Fermi-liquid ones.

The conserved disordered FM case is even more dramatic. Physically this is because a conservation law is equivalent to a long-ranged interaction (Hoyos and Vojta, 2007), and hence slows down relaxation even more. Nozadze and Vojta (2012) argued that in this case the relaxation time in \( d = 3 \) scales as

\[ \tau(L) \propto \exp[\alpha L^{3+n}], \]  

(3.97)

where \( n = 1 \) if the itinerant electrons are ballistic, and \( n = 2 \) if they are diffusive. Technically, the extra factor of \( L^n \) compared to the AFM (nonconserved) case is a result of the \( 1/|k|^n \) in the paramagnon propagator, Eq. (3.5a). Following the same steps as before, one can determine the physical observables. The local susceptibility behaves as

\[ \chi_{\text{loc}}^n(T \to 0) \propto \frac{1}{T} \exp[-A\{\ln(T_0/T)\}^{3/(3+n)}]. \]  

(3.98)

Here \( A \) is a constant and \( T_0 \) is a microscopic temperature scale. The specific heat is proportional to

\[ C_{\text{loc}}^n(T \to 0) \propto \exp[-A\{\ln(T_0/T)\}^{3/(3+n)}]. \]  

(3.99)

Finally, the magnetization \( m \) at zero temperature as a function of an applied field \( H \) is

\[ m(H \to 0) \propto \exp[-A\{\ln(H_0/H)\}^{3/(3+n)}], \]  

(3.100)

where \( H_0 \) is a microscopic magnetic field scale. Note that these exponentials go to zero slower than any power law.

2. Disordered local moments

A related concept in the presence of quenched disorder is that of local magnetic moments (Milovanovich, Sachdev, and Bhatt, 1989; Bhatt and Fisher, 1992). This topic was reviewed by Belitz and Kirkpatrick (1994). An important conclusion is that the interactions between the local moments, or rare regions, are very important.

3. Interacting rare regions

One conclusion of the previous section is that long-ranged RKKY interactions between local moments, in conjunction with rare-region effects, can have qualitative effects. This suggests that similar interactions between rare regions in a quantum Griffith phase might also be important. This question was studied by Dobrosavljević and Miranda (2005) and Case and Dobrosavljević (2007) for the case of a Heisenberg AFM. The applicability of these ideas, with suitable modifications, to FMs remains to be studied.

Dobrosavljević and Miranda (2005) considered rare regions centered at points \( R_i \) \( (i = 1, 2, 3, \ldots) \) that are characterized by local \( N \)-component \( (N > 1) \) OPs \( \phi_i(\tau) \), with \( \tau \) the imaginary-time variable. The Gaussian part of the action has the form

\[ S^{(2)}(\tau) = S_0^{(2)}(\tau) + S_{\text{int}}^{(2)}. \]  

(3.101)

Here \( S_0^{(2)}(\tau) \) is the noninteracting part,

\[ S_0^{(2)}(\tau) = \sum_{n,i} \int_0^\beta d\tau d\tau' \phi_i(\tau) \Gamma_n(\tau - \tau') \cdot \phi_i(\tau'). \]  

(3.102)

with \( \Omega_n \) a bosonic Matsubara frequency. Let us assume for simplicity that the OP is not conserved, so that the non-interacting vertex is given by

\[ \Gamma_0(\Omega_n) = \Gamma_0(0) + |\Omega_n|. \]  

(3.103)

The \( |\Omega_n| \) nonanalyticity is the Landau-damping mechanism due to the coupling of the magnetic OP to the conduction electrons that was discussed in Sec. III.A.1. In imaginary-time space, it corresponds to a power-law decay \( \Gamma_0(\tau \to \infty) \propto \tau^{-3/2} \). This puts the rare region or droplet, now considered a 1D classical system in \( \tau \) space with a 1/\( \tau^2 \) interaction, at its lower critical dimension (Joyce, 1969). This means that the non-interacting rare regions cannot develop long-range order.

The interacting part of the Gaussian action is given by

\[ S_{\text{int}}^{(2)} = \frac{1}{2} \sum_{n,i,j} \int_0^\beta d\tau d\tau' \phi_i(\tau) V(R_{ij}, \tau - \tau') \cdot \phi_j(\tau'). \]  

(3.104)

The interaction between two rare regions is assumed to be a static RKKY interaction given by

\[ V(R_{ij}, \tau) = \frac{J_{ij}}{(R_{ij})^3} \delta(\tau). \]  

(3.105)

\( J_{ij} \) is assumed to be a random amplitude of zero mean and variance \( (J_{ij}^2) = J^2 \). Using replica methods, Dobrosavljević and Miranda (2005) concluded that the effective contribution to the total action from rare-region interactions is

\[ \delta S = -\frac{1}{2} \sum_{i,j} \left( 1 - \delta_{ij} \right) \sum_{a,b} \frac{J_{ij}^2}{(R_{ij})^{2d}} \right) \]  

(3.106)

Here \( (\alpha, \beta) = 1, 2, \ldots, n \) are replica labels, and the replica limit \( n \to 0 \) is implied. Treating this interaction in a standard mean-field approximation gives

\[ \delta S = -\sum_{a,b} \sum_{n,i} \phi_i^a(\Omega_n) \Delta_{n,i}^{ab}(\Omega_n) \cdot \phi_i^b(-\Omega_n), \]  

(3.107)

where \( \Delta_{n,i}^{ab}(\Omega_n) \) is proportional to a weighted spatial average of a local rare-region susceptibility.
\[
\Delta^{\mathrm{gp}}(\Omega_n) = \frac{1}{N} \sum_{j \neq i} \frac{J^2}{|R_{ij}|^{2d}} \langle \phi_j^\alpha(\Omega_n) \cdot \phi_j^\beta(\Omega_n) \rangle. \tag{3.108}
\]

Within a self-consistent mean-field theory, the average in Eq. (3.108) is to be taken with respect to the complete action, including the rare-region interaction term.

Comparing Eqs. (3.102) and (3.107) we see that the rare-region interactions have renormalized the Gaussian part of the noninteracting action \( S_0 \). This is analogous to the effects of the fermionic soft modes discussed in Secs. III.B and III.C. The importance of this term depends on its behavior for long times or low frequencies. Dobrosavljevi \( \text{c} \) and Miranda (2005) concluded that effectively the noninteracting vertex \( \Gamma_0 \), Eq. (3.103), gets augmented by an additive term of the form

\[
\delta \Gamma(\Omega_n) \propto \text{const} + |\Omega_n|^{\alpha-1}. \tag{3.109}
\]

Here \( \alpha = c/\tilde{\sigma} \) is the same exponent that appears in Eqs. (3.95b) and (3.96). It is nonuniversal and is expected to decrease as the magnetically ordered phase is approached. Once \( \alpha < 0 \), the nonanalyticity coming from the rare-region interaction is stronger than the one due to Landau damping in the bare action, Eq. (3.103). The OP correlation function then falls off more slowly than \( 1/r^2 \) for large imaginary times.

The rare region thus is above its lower critical dimension and can develop long-range order. This in turn implies that sufficiently large droplets will freeze and form a cluster-glass phase (see footnote 19). This concept was used to analyze and predict the appearance of arbitrarily large, but exponentially rare, trap-free regions. This work left open the size of the asymptotic region. After many earlier studies, Barkema, Biswas, and van Beijeren (1982) and Kayser and Hubbard (1983), who showed that the asymptotic long-time behavior is dominated by the existence of arbitrarily large, but exponentially rare, trap-free regions. This work left open the size of the asymptotic region. After many earlier studies, Barkema, Biswas, and van Beijeren (1982) and Kayser and Hubbard (1983), who showed that the asymptotic long-time behavior is dominated by the existence of arbitrarily large, but exponentially rare, trap-free regions. This work left open the size of the asymptotic region. After many earlier studies, Barkema, Biswas, and van Beijeren (1982) and Kayser and Hubbard (1983), who showed that the asymptotic long-time behavior is dominated by the existence of arbitrarily large, but exponentially rare, trap-free regions.

The Griffiths phase of the classical bond-diluted Ising model mentioned in Sec. III.D.1 was also studied. The time-dependent local spin-spin correlation function \( C(t) \) is predicted to decay as \( (Bray, 1988, 1989) \)

\[
C(t \to \infty) \sim \exp[-\text{const} \times (\ln t)^{d/(d-1)}]. \tag{3.111}
\]

Monte Carlo simulations for \( d = 3 \) (Colborne and Bray, 1989) showed poor agreement with Eq. (3.111). Plotting \( \ln C(t) \) against \( (\ln t)^{1/2} \) yielded substantial curvature. A better fit was obtained by plotting \( \ln C(t) \) against \( (\ln(t/r))^{3/2} \), with \( r(T) \) an adjustable parameter. A still better fit was found using a stretched exponential or Kohlrausch form \( C(t) \sim \exp[-(t/r)^{\beta}] \), with \( \beta \) an increasing function of temperature that is on the order of 0.4. Various others (Colborne and Bray, 1989; Jain, 1995; Cao et al., 2006) found that \( C(t) \) must be less than \( 10^{-4} \) of its initial value before the asymptotic behavior sets in.

The situation is different for classical \( n \)-dimensional spins with \( n \geq 2 \). In this case, the Griffiths arguments predict (Bray, 1987, 1988, 1989)

\[
C(t \to \infty) \sim \exp[-\text{const} \times t^{1/2}]. \tag{3.112}
\]

Monte Carlo data are entirely consistent with this prediction for all but short times (Colborne and Bray, 1989).

For quantum systems, the increasing power of numerical methods has yielded interesting results. For a transverse-field Ising spin glass, Monte Carlo simulations on 2D and 3D systems by Guo, Bhatt, and Huse (1996) and Rieger and Young (1996) found clear evidence of Griffiths-phase effects. The size of the effects decreased by about a factor of 4 from \( d = 2 \) to 3. The strength of the effects, compared with classical systems, is sometimes attributed to the fact that in the quantum case the Griffiths clusters occur as line defects, as
opposed to point defects in classical models. Perhaps more importantly, because quantum tunneling of a rare region is a slower process \(\tau \propto \exp(L^d)\) than thermally activated dynamics \(\tau \lesssim \exp(L^{d-1})\) of the same rare region, the Griffiths singularities in the quantum case lead to power-law decays in time, or power-law singularities at low \(T\). These power-law effects in temperature can dominate the usual Fermi-liquid power laws in metals. Various susceptibilities may even diverge as \(T \to 0\).

Collectively, these results imply that the importance of the Griffiths effects is not a priori clear and may strongly depend on the nature of the system. For instance, in the classical case there is a qualitative difference between Ising and XY or Heisenberg models; see Eqs. (3.111) and (3.112). The quantum FM case, for both Ising and Heisenberg symmetry, is similar to the classical Ising model in the sense that there is an activation barrier to transport, unlike the classical Heisenberg case. On the other hand, there is numerical evidence for quantum mechanics enhancing the Griffiths effects.

### E. Textured phases as a way to avoid a quantum-critical point

Various others realized that the instability of Hertz theory can signalize either a first-order transition, or a transition into a nonhomogeneous phase (Belitz, Kirkpatrick, and Vojta, 1997; Chubukov, Pépin, and Rech, 2004; Rech, Pépin, and Chubukov, 2006). The respective conditions were investigated by several others (Maslov, Chubukov, and Saha, 2006; Efremov, Betouras, and Chubukov, 2008). Maslov and Chubukov (2009) concluded that in a model with a long-ranged exchange interaction the first-order transition always preempts the formation of a spiral phase.

Conduit, Green, and Simons (2009) used a self-consistent many-body approach supplemented by a numerical evaluation of fluctuation corrections to the free energy to argue that a spiral state can preempt the first-order transition as the FM state is approached from the PM phase. This textured magnetic phase is analogous to the Fulde-Ferrell-Larkin-Ovchinnikov state in superconductors (Fulde and Ferrell, 1964; Larkin and Ovchinnikov, 1964). Karahasanovic, Krüger, and Green (2012) expanded this to a purely analytical theory that allows for instabilities toward spin-nematic phases in addition to a spiral one. They proposed a phase diagram, Fig. 39, where upon approaching from the PM at low \(T\) one first encounters a spin-nematic phase, followed by a spiral phase, and finally a uniform FM. The possibility of a Pomeranchuk instability toward a non-s-wave ferromagnet or magnetic nematic had also been discussed earlier by Chubukov and Maslov (2009). Later work concluded that an infinite resummation of fluctuation contributions to the free energy results in the spiral phase occupying a substantially smaller part of the phase diagram (within about 1% of the transition point \(T = 0\)) than the original theory predicted (Pedder, Krüger, and Green, 2013), but the topology of the phase diagram remained the same. Such a narrow slice of spiral order would be easy to overlook experimentally and has so far not been observed. In 2D the theory predicts a much larger spiral phase. One must keep in mind, however, that no true long-range FM order is possible in \(d = 2\) at \(T > 0\).

![FIG. 39. Proposed phase diagram for a model allowing for spiral and spin-nematic order. \(\mu\) is the chemical potential, and \(g\) is the exchange coupling. From Karahasanovic, Krüger, and Green, 2012.](image)

### F. Other mechanisms for a first-order transition

The mechanism for a first-order transition in clean quantum FMs that was discussed in Sec. III.B.2 is remarkable because of its universality. However, in any given material a less universal mechanism may be present that by itself would suffice to drive the transition first order. Here we briefly discuss two such mechanisms.

#### 1. Band-structure effects

The coefficients in the Landau free energy

\[
f_L[m] = m^2 + u_4 m^4 + u_6 m^6 + O(m^8)
\]

depend in complicated ways on the microscopic details of the system, and, in particular, on band structure. In any given material it is possible that band-structure effects lead to a negative value of \(u_4\). If \(u_6 > 0\), this leads to a first-order transition at some positive value of \(t\), which preempts the second-order transition at \(t = 0\). Under certain conditions, correlations can have the same effect (Yamada, 1993). However, this cannot explain the universality of the observed effect in clean low-\(T\) FMs that is displayed by Tables I and II.

It is interesting that UGe\(_2\), in addition to the pressure-induced first-order PM-to-FM transition at \(p \approx 16\) kbar, shows a metamagnetic transition at a lower pressure that is also of first order. This transition, as well as the superconductivity that coexists with the ferromagnetism at low \(T\) and intermediate pressures, has been attributed to a special feature in the density of states of UGe\(_2\) (Pfleiderer and Huxley, 2002; Sandeman, Lonzarich, and Schofield, 2003; Shick et al., 2004a, 2004b).

#### 2. Magnetoelastic effects

 Phonons are generic soft modes in the sense of Sec. III.A.1 that couple to the magnetization. This can lead to a weakly first-order transition in classical magnets (Rice, 1954; Bean and Rodbell, 1962; Larkin and Pikin, 1969; Sak, 1974; Wegner, 1974; Bergman and Halperin, 1976; de Moura et al., 1976). We briefly review the conclusions...
for classical magnets and then the relevance of these results for quantum FM.

a. Classical magnets

Consider an LGW theory for a FM with OP \( M \) that couples to harmonic elastic degrees of freedom. In the simplest case of an isotropic 3D system in the continuum limit the action reads (Aharony, 1976)

\[
S = \int dx \left[ tM^2 + (\nabla M)^2 + u_4 M^4 + \left( \frac{K}{2} - \mu \right) \sum_{\alpha=1}^{3} u_{\alpha\alpha} \right]^{2} + \mu \sum_{\alpha,\beta} u_{\alpha\beta} + gM^2 \sum_{\alpha} u_{\alpha\alpha} \right].
\]  

(3.114)

Here \( K \) and \( \mu \) are elastic coefficients, and

\[
u_{\alpha\beta} = \frac{1}{2} \left( \partial_{\alpha} u_{\beta} + \partial_{\beta} u_{\alpha} + \sum_{\gamma} \partial_{\gamma} u_{\alpha} \partial_{\gamma} u_{\beta} \right)
\]

(3.115)

is the strain tensor in terms of derivatives of the displacement vector \( u(x) \). \( g \) is the magnetoelastic coupling constant. In systems on a lattice there are additional terms (Bergman and Halperin, 1976; de Moura et al., 1976), but the general structure of the action is the same. At constant pressure, additional terms coupling the pressure to the strain tensor need to be added (Imry, 1974).

There are several important features of this action. First, the coupling is to the square of the OP, \( u_{\alpha\alpha} \), which leads to a nonanalytic dependence of the free energy on the OP in a renormalized Landau theory. Here, by contrast, the coupling is much weaker due to the additional gradient, and the net effects of the elastic modes are additional terms of quartic order in the OP. Schematically, one can see this by replacing the strain tensor \( u_{\alpha\beta} \) by a scalar \( \epsilon \) and considering a Landau free energy

\[
f[m, \epsilon] = tm^2 + u_4 m^4 + Ke^2 + gm^2 \epsilon.
\]

(3.116)

Decoupling \( m \) and \( \epsilon \) shows that the transition in mean-field approximation is first order if \( g^2 > 4Ku_4 \). The nature of the phase transition as described by the LGW action (3.114) and its generalizations was studied by de Moura et al. (1976), who integrated out the elastic degrees of freedom, and by Bergman and Halperin (1976), who performed an RG analysis of the full coupled theory. The conclusion is consistent with the simple argument given above: For a sufficiently large magnetoelastic coupling the transition may become first order, but whether or not this occurs depends on the bare values of the parameters in the LGW theory, i.e., on microscopic details, as well as on the dimensionality of the OP (Nattermann, 1977). Magnetoelastic effects are a route to a first-order transition but not a universal route.

b. Quantum magnets

Gehring (2008) [see also Gehring and Ahmed (2010)] and Mineev (2011) proposed to apply these results for classical magnets to the quantum FM transition in pressure-driven systems by generalizing the Landau free energy (3.116) to

\[
f[m, \epsilon] = t(\epsilon)m^2 + u_4 m^4 + Ke^2
\]

with \( t(\epsilon) = T - T_c(\epsilon) \) representing the dependence of \( T_c \) on the strain (or, equivalently, on the pressure \( p \)). Expanding \( T_c(\epsilon) \) for small \( \epsilon \) leads to the coupling given in Eq. (3.116) with \( g \propto dT_c/dp \). Since experimentally one finds \( dT_c/dp \to \infty \) as \( T_c \to 0 \), they argued that effectively the magnetoelastic coupling \( g \) increases without bounds as \( T_c \) decreases, necessarily leading to a first-order transition at sufficiently low \( T_c \).

This line of reasoning is problematic. First, a singular dependence of a coefficient on a field must not be built into a Landau theory if the theory is to have any predictive value. Such a singular dependence may result from integrating out soft modes, such as in the treatment of classical liquid crystals or superconductors by Halperin, Lubensky, and Ma (1974), or in the renormalized Landau theory reviewed in Sec. III.B. In the case of compressible magnets such a result is implausible. The coupling between the magnetic OP and the elastic deformations is weak even in the classical case, see above, and in the quantum case it will be even weaker due to an additional frequency integral. Second, a diverging effective magnetoelastic coupling results in a diverging volume change (Bean and Rodbell, 1962). Therefore, even if one accepts the substitution of the observed \( T_c(p) \) into the Landau theory, it predicts that a structural phase transition must necessarily accompany the first-order magnetic transition. There is no experimental evidence for this. We conclude that currently no convincing theory for magnetoelastic effects in the quantum regime exists.

IV. SUMMARY, DISCUSSION, AND OUTLOOK

A. Summary and discussion

We have given an overview of the quantum phase-transition problem in metallic ferromagnets. Experimentally, a variety of phase diagrams are observed; see Fig. 2. Apart from discontinuous (first-order) and continuous (second-order) QPTs FM to a PM, a QPT from a FM state to an AFM or spin-wave state is observed in some systems, while in others the low-\( T \) phase near the onset of FM is some sort of a magnetic glass. In many systems with quenched disorder there is evidence for quantum Griffiths effects on the PM side of the transition. The experimental results are described in Sec. II, organized with respect to the structure of the phase diagram.

Theoretically, the transition from a PM quantum FM is expected to be discontinuous in clean systems and continuous in disordered ones. In either case the behavior at the QPT is very different from the one expected from Hertz theory. This is because of a coupling between the magnetization and soft fermionic excitations in metals that was included in Hertz theory in too simple an approximation and treated more thoroughly in the theory originally developed by two of us and T. Vojta that is reviewed in Sec. III. The results obtained
by Moriya, Hertz, andMillis are still expected to be observable in certain preasymptotic regimes. The agreement between these theoretical predictions and experimental results is generally very good for clean systems. Strongly disordered systems are much more complicated. Although the critical singularities at the continuous quantum FM transition have been calculated exactly, Griffiths-phase effects coexist with the critical singularities and complicate the experimental analysis.

We now add some remarks to the discussion in the main text and mention some related topics that we did not cover. The references in this section are intended to be illustrative, rather than exhaustive.

1. Nematic phases and transitions in a Fermi liquid have been investigated theoretically by Oganesyan, Kivelson, and Fradkin (2001). They used a Hertz-type theory, which yields a continuous transition with mean-field critical behavior in spatial dimensions $d = 2, 3$ for all types of nematics considered. The case of a metallic spin nematic, or non-$s$-wave FM, is theoretically closely related to the FM one. For such systems in the absence of quenched disorder it was later shown that the same mechanism operative in FMs generically causes the spin-nematic transition to be of first order (Kirkpatrick and Belitz, 2011).

For charge nematics, the mechanism leading to a first-order transition does not apply (Belitz, Kirkpatrick, and Vojta, 2002). Still, later work showed that the Hertz approach breaks down even in this case, but the breakdown is less dramatic than in the spin channel and the transition is believed to remain continuous (Dell’Anna and Metzner, 2006; Lee, 2009; Metliski and Sachdev, 2010).

There is experimental evidence of charge Ising-nematic order in systems including the pnictides (Chuang et al., 2010), Sr$_3$Ru$_2$O$_7$ (Borzi et al., 2007), and the normal state of the cuprates, in particular, YBa$_2$Cu$_3$O$_y$ (Daou et al., 2010). For a review, see Fradkin et al. (2010).

2. Another point is related to the models used to theoretically study the FM QPT. Hertz (1976) considered a continuum model of free electrons that interacts via a pointlike spin-triplet interaction. There are good reasons to believe that such a model does not actually have a FM phase; see Sec. IV.B.4. However, the point of an effective field theory such as Hertz’s is not to establish whether or not there is a phase transition in this, or any, model; it is to describe the properties of the transition, provided one actually occurs. The complicated band structure and other microscopic details that may well be necessary to produce a transition in the first place do not affect the universal properties at the transition and therefore can safely be omitted from the effective theory.

More recent theories [see, e.g., Kirkpatrick and Belitz (2012a)] considered an effective OP theory that has the existence of a magnetic transition encoded in the parameters of the effective LGW functional. All details of the solid-state structure that are necessary for FM to occur are thus hidden in these parameters. The OP is then coupled to fermions, and for capturing the qualitative effects of the latter on the FM transition again a simple continuum model suffices.

3. The near-universal observation of a first-order QPT in clean FMs is surprising even given the robustness of the effect discussed in Sec. III.B.2, since the term in the renormalized Landau theory that is responsible for it is logarithmic, which results in an exponential dependence of observables on parameters. It is possible that, perhaps as a result of strong electron correlations, an analog of van der Waals’s law of corresponding states for classical liquids holds for strongly correlated Fermi liquids, making the relevant parameters, measured in natural units, roughly the same in different materials. This notion is consistent with the discussion in Sec. II.B.5, and especially with the fact that the tricritical temperature scales roughly with the magnetic moment.

4. In Secs. II and III we emphasized the fact that experimental observations of continuous FM QPTs in strongly disordered systems are often difficult to interpret, and the critical exponents that characterize these transitions are hard to measure. However, qualitative features of both theoretical results and experimental observations indicate that several exponents are drastically different from both the mean-field exponents expected in a preasymptotic regime in weakly disordered systems and classical exponents in common universality classes.

For instance, the OP exponent $\beta$ is predicted to be larger than unity (about 1.2) in the preasymptotic regime where an effective power-law behavior is expected, whereas the exponent $\delta$ is unusually small (about 1.8); see Eq. (3.90c). In contrast, the mean-field values are $\beta = 1/2$ and $\delta = 3$, and the corresponding classical values for 3D Heisenberg FMs are about 0.37 and 4.8, respectively. Experiments do indeed tend to find values of $\beta$ and $\delta$ that are larger and smaller, respectively, than their respective mean-field values; see Secs. II.C.2 and III.C.3.h. A related issue is the shape of the phase boundary near the QPT, with both theory and experiments finding a tail in the phase diagram; see the discussion in Sec. III.C.3.h. Griffiths effects may also contribute to the observed properties in this region, which makes more detailed investigations desirable.

This superposition of critical phenomena and additional disorder effects notwithstanding, the results reviewed in Sec. III.C.3.e for the critical behavior of an FM OP coupled to diffusive fermions are believed to be exact. This type of problem also appears elsewhere. For instance, Savary, Moon, and Balents (2014) considered a model for pyrochlore iridates that couples a quantum $p^3$ theory to (in this case exotic) fermions, which results in a phase transition with similarly unusual critical properties.

5. Even far away from any QPT FM metals at low $T$ have interesting properties. This is not as well appreciated as the problems posed by AFMs, or by FMs near a QPT. For instance, in many clean FMs a generic
(i.e., existing in an entire phase) non-Fermi-liquid $T^{3/2}$ resistivity is observed over a large low-$T$ range in both the FM and PM phases (Sato, 1975; Pfleiderer et al., 2001; Niklowitz et al., 2005; Takahama et al., 2007; Brando et al., 2008). This is not well understood; see Sec. IV.B.

In disordered systems, Griffiths effects lead to generic NFL behavior on the PM side of the phase boundary as was discussed in Sec. III.D. In either phase, weak-localization (Lee and Ramakrishnan, 1985) and Altshuler-Aronov (Altshuler and Aronov, 1984), effects are expected in disordered systems. The resulting superimposed temperature dependences of observables can be quite intricate (Butenko, Bol’shtukhin, and Percherskaya, 1990) but in general little attention has been paid to them.

There has been interesting work on FM transitions in metals under nonequilibrium conditions (Mitra et al., 2006; Mitra and Millis, 2008), where correlations are generally greatly enhanced compared to systems in equilibrium (Belitz, Kirkpatrick, and Vojta, 2005).

In these systems the fermionic soft modes discussed in Sec. III are suppressed by boundary effects. As a consequence a Hertz-type nonequilibrium transition has been predicted.

(7) Unusual phases are expected in systems where both electronic correlations and a strong spin-orbit interaction are present (Wan et al., 2011). In particular, topological semimetal phases can occur which may be realized in $Y_3Ir_2O_7$ (Wan et al., 2011), $Bi_2Se_3$ (Zhang et al., 2009), and $HgCr_2Se_4$ (Xu et al., 2011), or in heterostructures of topological and normal insulators (Burkov and Balents, 2011). This semimetal state is a 3D analog of graphene and provides a condensed-matter realization of Weyl fermions. Calculations based on the LSDA + U + SO method (local spin-density approximation plus correlations plus spin-orbit coupling) have suggested a rich phase diagram with a QPT between a FM metal and a Weyl semimetal (Wan et al., 2011).

The nature of this transition has not been investigated.

Weyl semimetals also have interesting properties apart from any QPT. Ideas associated with them have been used to understand the intrinsic anomalous Hall effect in metallic FMs (Chen, Bergman, and Burkov, 2013). They argued that even Weyl nodes that do not coincide with the Fermi energy, as is believed to be the case in SrRuO$_3$, contribute to the intrinsic anomalous Hall conductivity in FM metals. This in turn implies that this conductivity in FMs is not purely a Fermi-surface property, which contradicts earlier conclusions (Haldane, 2004).

(8) FM transitions have been observed in a variety of quantum Hall systems. For instance, in a GaAs system in a perpendicular magnetic field, Piazza et al. (1999) observed a first-order transition in the $\nu = 2$ and 4 quantum Hall states. They suggested that the source of the observed hysteresis effects was not exotic, but was due to the expected domain structure in an easy-axis FM. Similar behavior was observed by De Poortere, Tutuc, and Shayegan (2003) in AlAs quantum wells. Drichko et al. (2012) measured magneto-resistance properties in two $p$-Si/SiGe/Si quantum-well samples in a tilted magnetic field. In a sample with $p = 2 \times 10^{11}$ cm$^{-2}$ they observed phase coexistence and concluded that there was a first-order FM-PM transition. However, in the second sample with $p = 7.2 \times 10^{10}$ cm$^{-2}$ no transition was observed. Stoner or RPA-like theories have been used to discuss FM transitions in quantum Hall systems (Burkov and MacDonald, 2002; Lopatnikova, Simon, and Demler, 2004), and for the pseudospin FM realized in bilayer quantum Hall systems there is evidence for a first-order transition (Schliemann, Girvin, and MacDonald, 2001; Zou et al., 2010; Lee et al., 2014).

B. Open problems

We finally mention some open problems.

(1) Additional work is needed to disentangle Griffiths singularities and critical singularities near the FM QPT in disordered metals. Since Griffiths singularities generically are stronger on the PM side of the transition (Motrunich et al., 2000), the QPT is best studied from the FM side. Although Griffiths singularities exist in weaker forms also on the FM side, the existence of a zero-field magnetization uniquely implies long-ranged FM order, so the singular behavior of the zero-field magnetization itself can distinguish between Griffiths singularities and critical singularities. The relation between Griffiths physics and the Harris criterion was discussed by Vojta and Hoyos (2014) and Vojta, Igo, and Hoyos (2014).

The NFL behavior observed in many clean materials in large parts of the phase diagram was reviewed by Stewart (2001) and remains incompletely understood. One manifestation is the $T^{3/2}$ behavior of the resistivity that was mentioned in Sec. IV.A. An explanation in terms of columnar fluctuations, which is applicable to MnSi, was proposed by Kirkpatrick and Belitz (2010). However, because of the large variety of materials where a $T^{3/2}$ resistivity is observed, it is likely that more than one mechanism can lead to this behavior. For a related discussion of ZrZn$_2$, see Smith et al. (2008).

Similarly, weak-localization and Altshuler-Aronov effects in weakly disordered FMs deserve more attention. The $T$ dependence of the resistivity can be complicated, with many contributions from very different sources (Mizutani et al., 1988; Butenko, Bol’shtukhin, and Percherskaya, 1990; Yildiz et al., 2009).

(2) In the presence of magnetic impurities, or impurities with a large spin-orbit coupling, the soft fermionic modes in the disordered case will be suppressed (Lee and Ramakrishnan, 1985; Belitz and Kirkpatrick, 1994), and the nature of the FM QPT is unclear. It is possible that, once the generic soft modes have been eliminated, the transition will resemble the one in disordered AFM metals, but not much is known about this case.
(3) There are materials in which no FM transition has been observed, but that nonetheless display very interesting properties. One of these is YFe$_2$Al$_{10}$. It crystallizes in the eponymous orthorhombic structure with a single Fe site (Kerkau et al., 2012). Initial experiments identified correlated FM behavior (Strydom and Peratheepan, 2010). Further detailed studies on single crystals found anomalies in the magnetic susceptibility and the specific heat which obey a peculiar NFL field-temperature scaling (Park et al., 2011; Wu et al., 2014). In addition, FM correlations have been found in NMR experiments (Khuntia et al., 2012). These observations have been interpreted as indicating that the material is close to a FM QPT. However, no FM transition has been detected so far at temperatures down to 50 mK, even upon doping with a small surplus of Fe (Strydom et al., 2013). On the contrary, Fe excess or deficiency drive YFe$_2$Al$_{10}$ away from the critical behavior. The low-T resistivity shows a Kondo-like logarithmic increase below 30 K with a high $\rho_0 \approx 75 \ \mu\Omega\text{cm}$ (RRR = 2), which puts YFe$_2$Al$_{10}$ in the group of strongly disordered systems (cf. Sec. II.C.2). However, single-crystal structure refinement did not find any deviation from the ideal composition (Kerkau et al., 2012), so the origin of the large resistivity is not clear. The observed scaling behavior and the lack of a magnetically ordered phase in YFe$_2$Al$_{10}$ need further investigation. We also mention that in the closely related system YbFe$_2$Al$_{10}$ strong FM correlations have been observed at low $T$ (Khuntia et al., 2014). In this material, the Yb-derived electrons at low $T$ form a nonmagnetic intermediate-valent state and therefore the Fe atoms alone are responsible for the FM correlations, as is the case in YFe$_2$Al$_{10}$.

(4) There are materials that display a transition from a metallic AFM state to a FM at low $T$. Two examples are CeRu$_2$Ge$_2$ (Raymond, Haen et al., 1999) and CeRu$_2$Al$_2$B (Baumbach et al., 2012). In both cases, the AFM-FM transition is first order, whereas the transition from a PM to an AFM at a higher Néel temperature is second order. It is plausible that the QPT from a metallic AFM to a FM in clean systems is first order for the same reasons as that from a metallic PM to a FM, but no theory is available for this case.

A related issue is the detailed structure of the phase diagrams discussed in Sec. II.D. These systems all must display a Lifshitz point and at least two QPTs. In clean systems, the QPT from the FM phase to the modulated phase is expected to be first order, but this needs experimental confirmation. In disordered systems, it may well be a novel type of QCP. Similarly, the Lifshitz point may be a multicritical point with very interesting properties.

(5) An old question is what ingredients in a model are necessary for producing itinerant FM [see, e.g., Varma (2010) and Shimizu (1964), and references therein]. It has long been suspected that in simple electron-fluid models there is no FM phase (Ceperley and Alder, 1980; Chang, Zhang, and Ceperley, 2010), although some recent quantum Monte Carlo studies suggest otherwise (Pilati et al., 2010; Pilati, Zilchenko, and Troyer, 2014). This topic has received much attention recently in the context of optical lattices, especially an experiment that reported itinerant FM in a Fermi gas of ultracold atoms (Jot et al., 2009). However, subsequent experiments by the same group cast doubt on the original interpretation of the data (Santer et al., 2012). FM solid-state systems typically have a complicated band structure. Whether or not FM can occur in optical lattices is an open question. If it does, the transition is expected to be first order for the same reasons as in clean solid-state systems (Duine and MacDonald, 2005), and a quantum Monte Carlo study of a 2D Stoner Hamiltonian suggests that the strength of the first-order transition depends on the range of the interaction (Conduit, 2013).

(6) Quenches, i.e., rapid changes of external parameter values, at $T = 0$ in both clean and disordered metallic FMs are interesting. Belitz, Kirkpatrick, and Saha (2007) showed that the coupling of the OP to the fermionic soft modes leads to qualitatively new effects for the late-stage coarsening. Gagel, Orth, and Schmalian (2014) showed that there is universal preasymptotic behavior in general quantum quench problems due to long-range boundary effects. In FM metals this effect is expected to be even more interesting because of the coupling to the fermionic soft modes.

(7) The experimental coexistence curve appears to be extremely steep in many FM systems [see, e.g., Figs. 5 and 10, and the phase diagrams for ZrZn$_2$ measured by Uhlbrz, Pfeiderer, and Hayden (2004) and Takashima et al. (2007)], but determining the coexistence curve from different observables can lead to different detailed shapes (Kabeya et al., 2010). Studies of the detailed shape, by pressure cycling in the $p-T$ plane, or field cycling in the $p-H$ plane, would be interesting. Theoretically, the shape of the coexistence curve can be determined from the Clapeyron-Clausius equation, which has been discussed for quantum Hall systems by Zou et al. (2010) and for QPTs in general and FMs, in particular, by Kirkpatrick and Belitz (2015b).

(8) Without trying to be exhaustive, we mention a few other FM materials that may be candidates for suppressing $T_C$ via pressure or chemical substitution: NpNiSi$_2$, a Kondo-lattice system with $T_C = 51.5$ K (Colineau et al., 2008); Sr$_x$Ru$_2$O$_{10}$, a layered FM with $T_C = 148$ K (Cao et al., 1997; Crawford et al., 2002) (see also Sec. II.B.4); the enhanced PM TiBe$_2$ which shows metamagnetism at 5 T (Wohlfarth, 1980); and TiBe$_2$-Cu$_y$ which shows a transition to a FM ordered state (Giori et al., 1979; Acker et al., 1981). The latter system was intensively investigated in the early 1980s, but a detailed and conclusive phase diagram does not exist. Since recent band-structure calculations (Jeong, Kyker, and Pickett, 2006) suggest that TiBe$_2$ is close to an AFM instability, it would be interesting to revisit the phase diagram of TiBe$_2$-$y$Cu$_y$. 
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APPENDIX A: LIST OF ACRONYMS

AFM Antiferromagnet, or antiferromagnetism, or antiferromagnetic
CDW Charge-density wave
CEF Crystalline electric field
CEP Critical end point
DIV Dangerous irrelevant variable
FM Ferromagnet, or ferromagnetism, ferromagnetic
LGW Landau-Ginzburg-Wilson
NFL Non-Fermi liquid
PM Paramagnet, or paramagnetic
RG Renormalization group
QCP Quantum critical point
QCEP Quantum critical end point
QPT Quantum phase transition
RGR Renormalization group
RRR Residual resistance ratio

APPENDIX B: DEFINITIONS OF CRITICAL EXPONENTS

Let $T$ be the temperature, $t$ the dimensionless distance from criticality at $T = 0$, and $h$ the magnetic field. Consider the correlation length $\xi$, the magnetization $m$, the magnetic susceptibility $\chi$, and the specific-heat coefficient $\gamma = C/T$ as functions of $t$, $T$, and $h$, and the susceptibility also as a function of the wave number $k$. We define critical exponents as follows.

Correlation length:

$$\xi(t \to 0, T = 0) \propto |t|^{-\nu},$$
$$\xi(t = 0, T \to 0) \propto T^{-\nu/\delta}.$$  

Order parameter:

$$m(t \to 0, T = 0, h = 0) \propto (-t)^{\beta},$$
$$m(t = 0, T \to 0, h = 0) \propto T^{\beta \gamma},$$
$$m(t = 0, T = 0, h \to 0) \propto h^{1/\delta}.$$  

Order-parameter susceptibility:

$$\chi(t \to 0, T = 0; k = 0) \propto |t|^{-\gamma},$$
$$\chi(t = 0, T \to 0; k = 0) \propto T^{-\nu},$$
$$\chi(t = 0, T = 0, k \to 0) \propto 1/k^{2-\eta}.$$  

Specific-heat coefficient:

$$\gamma(t \to 0, T = 0) \propto |t|^{-\alpha},$$
$$\gamma(t = 0, T \to 0) \propto T^{-\beta \gamma}.$$  

$\nu$, $\beta$, $\gamma$, $\delta$, and $\eta$ are defined in analogy to the corresponding exponents at a classical phase transition. The definition of $\alpha$ deviates from the one of the classical exponent customarily denoted by $\alpha$, which is defined in terms of the specific heat rather than the specific-heat coefficient. At a classical phase transition, $\alpha$ coincides with $\alpha_F$, $\alpha_T$, $\nu_T$, and $\gamma_T$ reflect the fact that a QPT can be approached either in the $T = 0$ plane, or from $T > 0$. The definition of $\beta_T$ in Eq. (B2) is purely formal; see Kirkpatrick and Belitz (2015a) and the discussion after Eq. (3.58).

REFERENCES