39. **Electromagnetic waves and gauge invariance**

a) Show that the Lorenz gauge, \( \frac{1}{c} \partial_t \varphi + \nabla \cdot \mathbf{A} = 0 \), still does not uniquely determine the potentials of an electromagnetic wave: Let \( f \) be an arbitrary scalar solution of the wave equation, \( \Box f = 0 \). Then the transformation \( \mathbf{A} \rightarrow \mathbf{A} + \nabla f \), \( \varphi \rightarrow \varphi - \frac{1}{c} \partial_t f \) leaves both the wave equation for the 4-vector potential and the fields unchanged.

b) Show in particular that the gauge of an electromagnetic wave can always be chosen such that \( \varphi = 0 \), \( \nabla \cdot \mathbf{A} = 0 \).

(3 points)

40. **Plane waves**

Consider the scalar field

\[ \psi(x, t) = \cos(k \cdot x - \omega t) , \]

where \( k \) is a Euclidian vector.

a) What is necessary and sufficient to make \( \psi \) a solution of the wave equation?

b) Perform a Lorentz boost, and show that the transformed wave again has the form

\[ \psi'(x', t') = \cos(k' \cdot x' - \omega' t') . \]

How are \( k' \) and \( \omega' \) related to \( k \) and \( \omega \)?

(3 points)

41. **Spherical waves**

Consider the wave equation

\[ \left( \frac{1}{c^2} \partial_t^2 - \nabla^2 \right) f(x, t) = 0 \]

Find and discuss the most general solution that has the form

\[ f(x, t) = u(r, t)/r \]

where \( r = |x| \).

(3 points)