5. **Basis transformations**

Consider $\mathbb{R}^n$ as a vector space with a basis $\{e_i\}$, and let $D$ be a basis transformation in the sense of §2.2. Show that the vectors

$$\tilde{e}_i = e_j (D^{-1})^j_i$$

do indeed form a basis. (3 points)

6. **Transformation of the metric**

Let $\{e_i\}$ be a basis on $\mathbb{R}^n$, and let $g$ be a metric ($g_{ij} = e_i \cdot e_j$). Let $D^{-1}$ be a basis transformation: $\tilde{e}_i = e_j (D^{-1})^j_i$. Show that the metric $\tilde{g}$ in the basis $\{\tilde{e}_i\}$ is given by

$$\tilde{g} = (D^{-1})^T g D^{-1}$$

and that the inverse of this relation is

$$g = D^T \tilde{g} D$$

(2 points)

7. **Normal coordinate transformations**

Prove the lemma and the theorem from §2.4. (3 points)

8. **Lorentz transformations**

Consider the 2-dimensional Minkowski space $M_2$ with metric

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $2 \times 2$ matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves $g$ invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix $J$ (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine $J$ explicitly. (6 points)