12. Dual field tensor
Prove the proposition from ch.2 §1.1, i.e., show that the dual field tensor \( \tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa} \) obeys
\[
\partial_\mu \tilde{F}^{\mu\nu}(x) = 0
\] (2 points)

13. Functional derivative
Let \( F[\varphi] \) be a functional of a real-valued function \( \varphi(x) \). For simplicity, let \( x \in \mathbb{R} \); the generalization to more than one dimension is straightforward. We can (sloppily) define the functional derivative of \( F \) as
\[
\delta F \quad \delta \varphi(x) := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( F[\varphi(y) + \epsilon \delta (y - x)] - F[\varphi(y)] \right)
\]
a) Calculate \( \delta F / \delta \varphi(x) \) for the following functionals:
   i) \( F = \int dx \varphi(x) \)
   ii) \( F = \int dx \varphi^2(x) \)
   iii) \( F = \int dx (\varphi'(x))^2 \) where \( \varphi'(x) = d\varphi/dx \)
      *hint*: Integrate by parts and assume that the boundary terms vanish.
   iv) \( F = \int dx V(\varphi'(x)) \) where \( V \) is some given function.

*remark*: Blindly ignore terms that formally vanish as \( \epsilon \to 0 \) unless you want to find out why the above definition is problematic. It does work for operational purposes, though.

b) Consider a “Lagrangian” \( \mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) \) (i.e., a function of \( \varphi \) and its derivatives) and an “action” \( S = \int d^4 x \mathcal{L} \). Show that extremizing \( S \) by requiring \( \delta S / \delta \varphi(x) \equiv 0 \) with the above definition of the functional derivative leads to the Euler-Lagrange equations
\[
\partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}
\] (3 points)

14. Massive scalar field
a) Consider a Lagrangian density
\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \varphi(x) \right) \left( \partial^\mu \varphi(x) \right) - \frac{m^2}{2} (\varphi(x))^2
\]
for a real scalar field \( \varphi(x) \). What is the Euler-Lagrange equation for the field \( \varphi \)?

b) Generalize this Lagrangian density to a complex field \( \phi(x) \in \mathbb{C} \):
\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \phi(x) \right) \left( \partial^\mu \phi^*(x) \right) - \frac{m^2}{2} |\phi(x)|^2
\]
What are the Euler-Lagrange equations now?

c) Consider a local gauge transformation, \( \phi(x) \to \phi(x) e^{i\Lambda(x)} \), with \( \Lambda(x) \) a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(3 points)
.../over
15. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

\[
S_{GL} = \int d\mathbf{x} \left[ r |\phi(\mathbf{x})|^2 + c \left( |\nabla - iq\mathbf{A}(\mathbf{x})| \phi(\mathbf{x})|^2 + u |\phi(x)|^4 + \frac{1}{16\pi\mu} F_{ij}(\mathbf{x}) F^{ij}(\mathbf{x}) \right) \right]
\]

Here \( \mathbf{x} \in \mathbb{R}^3 \), \( \phi(\mathbf{x}) \) is a complex-valued field that describes the superconducting matter, \( \mathbf{A} \) is the Euclidian vector field that comprises the spatial components of the 4-vector \( A^\mu = (A^0, \mathbf{A}) \), and \( F_{ij} = \partial_i A_j - \partial_j A_i \) \( (i,j = 1,2,3) \). \( \mu \) and \( q \) are coupling constants that characterize the vector potential and its coupling to the matter, and \( r, c \) and \( u \) are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action.

b) Show that this theory is invariant under gauge transformations \( \phi(\mathbf{x}) \rightarrow \phi(\mathbf{x}) e^{iq\lambda(\mathbf{x})}, \mathbf{A}(\mathbf{x}) \rightarrow \mathbf{A}(\mathbf{x}) + \nabla \lambda(\mathbf{x}) \).

c) Show that the Lorentz-invariant Lagrangian density from Problem 14 b) can be made gauge invariant by coupling \( \phi(\mathbf{x}) \) to the electromagnetic vector potential \( A^\mu(\mathbf{x}) \).

*hint:* Replace the 4-gradient \( \partial_\mu \) by \( D_\mu = \partial_\mu + iqA_\mu \) and add the Maxwell Lagrangian.

*note:* If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)