27. **1-d Fourier transforms**

Consider a function \( f \) of one real variable \( x \). Calculate the Fourier transforms of the following functions:

a) \( f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

b) \( f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

c) \( f(x) = e^{-(x/x_0)^2} \).

(3 points)

28. **3-d Fourier transforms**

Consider a function \( f \) of one vector variable \( x \in \mathbb{R}^3 \). The Fourier transform \( \hat{f} \) of \( f \) is defined as

\[
\hat{f}(k) = \int dx \ e^{-ik \cdot x} f(x).
\]

Calculate the Fourier transforms of the following functions:

a)

\[
f(x) = \begin{cases} 1 & \text{for } r < r_0 \\ 0 & \text{otherwise} \end{cases} \quad (r = |x|)
\]

b)

\[
f(x) = 1/r.
\]

**hint:** Consider \( g(x) = \frac{1}{r} e^{-r/r_0} \) and let \( r_0 \to \infty \).

(3 points)

29. **More 1-d Fourier transforms**

Consider a function of time \( f(t) \) and define its Fourier transform

\[
\hat{f}(\omega) := \int dt \ e^{i \omega t} f(t)
\]

and its Laplace transform \( F(z) \) as

\[
F(z) = \pm i \int dt \ e^{zt} f_\pm(t) \quad (\pm \text{ for } \text{sgn} (\text{Im } z) = \pm 1)
\]

with \( z \) a complex frequency and \( f_\pm(t) = \Theta(\pm t) f(t) \). Further define

\[
F''(\omega) = \frac{1}{2i} [F(\omega + i0) - F(\omega - i0)] \quad , \quad F'(\omega) = \frac{1}{2} [F(\omega + i0) + F(\omega - i0)]
\]

.../over
Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon \to 0} \epsilon / (x^2 + \epsilon^2) = \pi \delta(x)$, as we will show in Problem 31.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)