30. Regularizations of the constant function, and of the sign function

a) Prove the statement from ch.3 §2.4 example 1: The sequence \( f_n(x) = e^{-x^2/n^2} \) is a regular sequence of test functions that is a regularization of the generalized function \( f(x) \equiv 1 \).

b) Prove the statement from ch.3 §2.4 example 4: The sequence \( f_n(x) = \tanh(nx) \) is a regularization of the generalized function \( f(x) = \text{sgn} \, x \).

31. Distribution limits

a) Show that the sequences

\[
f_n(x) = \frac{1}{\pi} \sin(nx) \quad (n = 1, 2, \ldots)
\]

and

\[
g_n(x) = \frac{1}{\pi n} \frac{1}{x^2 + 1/n^2} \quad (n = 1, 2, \ldots)
\]

yield the \( \delta \)-function as \( n \to \infty \) in a distribution-limit sense:

\[
\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} g_n(x) = \delta(x)
\]

b) Show that

\[
\frac{d}{dx} \text{sgn} \, x = 2 \, \delta(x)
\]

and

\[
\frac{d}{dx} \Theta(x) = \delta(x)
\]

with \( \Theta(x) \) the step function.

c) Show that

\[
\lim_{\epsilon \to 0} \epsilon |x|^{\epsilon-1} = 2 \, \delta(x)
\]

*hint:* Start with ch.3 §2.4 example 6 and differentiate.

32. Derivatives of the \( \delta \)-function

Let \( \delta^{(n)}(x) \) be the \( n \)-th derivative of the \( \delta \)-function. Show that, for any weakly increasing function \( \phi(x) \),

\[
\delta^{(m)}(x) \phi(x) = \sum_{n=0}^{m} (-1)^n \binom{m}{n} \phi^{(n)}(0) \delta^{(m-n)}(x)
\]

(3 points)