9. **Pauli group**

The Pauli matrices are complex $2 \times 2$ matrices defined as

$$
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

Now consider the set $P_1$ that consists of the Pauli matrices and their products with the factors $-1$ and $\pm i$:

$$
P_1 = \{ \pm \sigma_0, \pm i\sigma_0, \pm \sigma_1, \pm i\sigma_1, \pm \sigma_2, \pm i\sigma_2, \pm \sigma_3, \pm i\sigma_3 \}
$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)

10. **The group $S_3$**

a) Compile the group table for the symmetric group $S_3$. Is $S_3$ abelian?

b) Find all subgroups of $S_3$. Which of these are abelian?

(6 points)

11. **Abelian groups**

Let $(G, \vee)$ be a group with neutral element $e$. Let $a \in G$ be a fixed element, and define a mapping $\varphi : G \to G$ by $\varphi(x) = a \vee x \vee a^{-1}$ $\forall x \in G$.

a) Show that $\varphi$ defines an automorphism on $G$, called an *inner automorphism*.

b) Show that abelian groups have no inner automorphisms except for the identity mapping $\varphi(x) = x$.

c) Let $g \vee g = e \forall g \in G$. Prove that $G$ is abelian.

(6 points)

12. **Fields**

a) Show that the set of rational numbers $\mathbb{Q}$ forms a commutative field under the ordinary addition and multiplication of numbers.

b) Consider a set $F$ with two elements, $F = \{\theta, e\}$. On $F$, define an operation “plus” (+), about which we assume nothing but the defining properties

$$
\theta + \theta = \theta, \quad \theta + e = e + \theta = e, \quad e + e = e
$$

Further, define a second operation “times” (·), about which we assume nothing but the defining properties

$$
\theta \cdot \theta = e \cdot \theta = \theta \cdot e = \theta, \quad e \cdot e = e
$$

Show that with these definitions (and no additional assumptions), $F$ is a field.

(7 points)