21. Lorentz transformations

Consider the 2-dimensional Minkowski space $M_2$ with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $2 \times 2$ matrix representations of the pseudo-orthogonal group $O(1, 1)$ that leaves $g$ invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1, 1)$ can be written in the form

$$D_{\sigma, \tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1, 1)$ it thus suffices to study the matrices $D(\phi) := D_{+1, +1} = \begin{pmatrix} \cosh \phi & \sinh \phi \\ \sinh \phi & \cosh \phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1, 1)$ that is sometimes denoted by $SO^+(1, 1)$), and that the mapping $\phi \rightarrow D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix $J$ (called the generator of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine $J$ explicitly.

(6 points)

22. Lindhard function

Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ (which plays an important role in the theory of many-electron systems) defined by

$$f(z) = \log \left( \frac{z - 1}{z + 1} \right)$$

The spectrum $f'' : \mathbb{R} \rightarrow \mathbb{R}$ and the reactive part $f' : \mathbb{R} \rightarrow \mathbb{R}$ of $f$ are defined by

$$f''(\omega) := \frac{1}{2i} [f(\omega + i0) - f(\omega - i0)] \quad , \quad f'(\omega) := \frac{1}{2} [f(\omega + i0) + f(\omega - i0)]$$

a) Show that $f'$ and $f''$ are indeed real-valued functions.

b) Determine $f''$ and $f'$ explicitly, and plot them for $-3 < \omega < 3$.

c) Show that

$$\int_\infty^\infty \frac{d\omega}{\pi} \frac{f''(\omega)}{\omega - z} = f(z)$$

(5 points)
23. Another causal function

The function considered in Problem 24. is an example of a class of complex functions called *causal functions* that are important for the theory of many-particle systems. Another member of this class is

$$g(z) = \sqrt{z^2 - 1} - z$$

Determine the spectrum and the reactive part of $g(z)$, and plot them for $-3 < \omega < 3$. (3 points)

24. Exponentials

Consider the exponential function

$$f(z) = e^z = e^{z'} + iz''$$

a) Show that $f(z)$ is analytic everywhere in $\mathbb{C}$.

b) Convince yourself explicitly that the real and imaginary parts of $f$ obey Laplace’s differential equation.

c) Show that $df/dz\big|\Delta = f(z)$.

d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \quad , \quad \sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$$

are analytic everywhere in $\mathbb{C}$, and that

$$\frac{d}{dz} \cos z = -\sin z \quad , \quad \frac{d}{dz} \sin z = \cos z.$$ 

(4 points)