25. Applications of the residue theorem
Use complex analysis to evaluate the real integrals

a) \[ \int_{-\infty}^{\infty} dx \frac{\sin x}{x} \]

\textit{hint:} Write \( \sin x = (e^{ix} - e^{-ix})/2i \) and consider the resulting two integrals with complex integrands. Why is this a good strategy?

b) \[ \int_{-\infty}^{\infty} dx \frac{\sin x}{x} \frac{1}{1 + x^2} \]

and check your results by means of Wolfram Alpha.

Let \( a \in \mathbb{C} \) with \( \Re a > 0 \). Use the residue theorem to show that

c) \[ \int_{-\infty}^{\infty} dx e^{-ax^2} = \sqrt{\pi/a} \]

Now let \( a \in \mathbb{R} \) and consider the integral
d) \[ \int_{-\infty}^{\infty} dx \frac{1}{x^2 + a^2} \]

and define its Cauchy principal value by

\[ \lim_{R \to 0} \left[ \int_{-\infty}^{R} dx f(x) + \int_{R}^{\infty} dx f(x) \right] \]

with \( f(x) = 1/x(x^2 + a^2) \). Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

\textit{hint:} Go around the pole on a semicircle of radius \( R \) and let \( R \to 0 \).

(11 points)

26. 1-d Fourier transforms
Consider a function \( f \) of one real variable \( x \). Calculate the Fourier transforms of the following functions:

a) \[ f(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

b) \[ f(x) = \begin{cases} 1 - |x| & \text{for } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases} \]

c) \[ f(x) = e^{-(x/x_0)^2} \]

(3 points)
27. 3-d Fourier transforms

Consider a function $f$ of one vector variable $x \in \mathbb{R}^3$. The Fourier transform $\hat{f}$ of $f$ is defined as

$$\hat{f}(k) = \int d\mathbf{x} \ e^{-i\mathbf{k} \cdot \mathbf{x}} f(\mathbf{x}) .$$

Calculate the Fourier transforms of the following functions:

a) $$f(x) = \begin{cases} 1 & \text{for } r < r_0 \\ 0 & \text{otherwise} \end{cases} \quad (r = |x|)$$

b) $$f(x) = \frac{1}{r} .$$

*hint* Consider $g(x) = \frac{1}{r} e^{-r/r_0}$ and let $r_0 \to \infty$.

(3 points)

28. More 1-d Fourier transforms

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt \ e^{i\omega t} f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm i \int dt \ e^{izt} f_\pm(t) \quad (\pm \text{ for } \text{sgn}(\text{Im } z) = \pm 1)$$

with $z$ a complex frequency and $f_\pm(t) = \Theta(\pm t) f(t)$. Further define

$$F''(\omega) = \frac{1}{2i} [F(\omega + i0) - F(\omega - i0)] , \quad F'(\omega) = \frac{1}{2} [F(\omega + i0) + F(\omega - i0)]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

*hint* $\lim_{\varepsilon \to 0} \varepsilon/(x^2 + \varepsilon^2) = \pi \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases $\int \frac{d\omega}{\pi} F''(\omega)/\omega = F'(\omega = 0)$.

*note* These concepts are important for the theory of response functions.

(4 points)