13. Function space
Consider the set $C$ of continuous functions $f : [0, 1] \rightarrow \mathbb{R}$. Show that by suitably defining an addition on $C$, and a multiplication with real numbers, one can make $C$ an additive vector space over $\mathbb{R}$.

(2 points)

14. The space of rank-2 tensors
Prove the theorem of ch.1 §4.3: If $x$, $y$ are elements of a vector space of dimension $n$, then the set of rank-2 tensors $t = x \otimes y$ forms a vector space of dimension $n^2$.

(3 points)

15. Cross product of 3-vectors
Let $x, y \in \mathbb{R}^3$ be vectors, and let $\epsilon_{ijk}$ be the Levi-Civita tensor. Show that the (covariant) components of the cross product $x \times y$ are given by

$$(x \times y)_i = \epsilon_{ijk} x^j y^k$$

(1 point)

16. Symmetric tensors
Let $V$ be an $n$-dimensional vector space over $K$ with some basis, let $f : V \times V \rightarrow K$ be a bilinear form, and let $t$ be the rank-2 tensor defined by $f$. Show that $f$ is symmetric, i.e. $f(x,y) = f(y,x) \forall x, y \in V$, if and only if the components of the tensor with respect to the given basis are symmetric, i.e., $t_{ij} = t_{ji}$.

(2 points)
[3.] On $C$, define $(f+g)(x) = f(x) + g(x)$

If $f$ and $g$ are continuous, then so is the sum and defined $(f+g)$ is

- Known ✓

Furthermore, via $f(x) \in R$, $C$ inherits all of the other group
properties from $(R, +)$

$\implies C$ is an additive group

Now define multiplication via scalars $\lambda \in R$ by $(\lambda f)(x) = \lambda f(x)$

If $f$ is continuous, then so is the product $(\lambda f)$.

Furthermore, via $\lambda \in R$ and $f(x) \in R$, this multiplication with
scalars is bilinear and associative, as it inherits these properties
from $R$ under ordinary addition and multiplication of numbers.

Finally, $(1f)(x) = 1f(x) = f(x) \forall x \in (0,1) \implies 1f = f$

$\implies C$ is a $R$-vector space.
we already know that the rank-2 norms to one-to-one

components to bilinear forms \( f(x, j) \).

On the set of bilinear forms, define an addition by

\[
(f + g)(x, j) = f(x, j) + g(x, j)
\]

This makes the set of \( \mathbb{R} \) an additive group.

Now define a multiplication with scales by

\[
(\lambda f)(x, j) = \lambda f(x, j), \quad \lambda \in \mathbb{R}
\]

Now the set of \( f \) is a \( \mathbb{R} \)-vector space. This is the derived

space of norms of rank 2.

Finally, any bilinear form

\[
f(x, j) = t_{ij} x^i x^j
\]

can be expanded as a linear combination of the \( t^j_{\cdot i} \)

components \( t_{ij} \), along with all \( \mathbb{R} \)-linear

combinations. The necessary for spanning the space of

norms \( \mathbb{R} \)-span has dimension \( \mathbb{R}^2 \).
15. Let $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$. The cross product is defined by

$$\mathbf{x} \times \mathbf{y} = (x^2 y^3 - x^3 y^2, x^3 y^1 - x^1 y^3, x^1 y^2 - x^2 y^1)$$

On the other hand,

$$\varepsilon_{ijk} \mathbf{x}^i \mathbf{y}^j \mathbf{y}^k = \begin{cases} 
  x^2 y^3 - x^3 y^2 & i = 3 \\
  x^3 y^1 - x^1 y^3 & i = 2 \\
  x^1 y^2 - x^2 y^1 & i = 1
\end{cases} = (\mathbf{x} \times \mathbf{y})^i$$
Let \( f(x, y) = f(y, x) \) for all \( x, y \in V \).

Then the transpose is defined by \( t_{ij} = f(e_i, e_j) \) for all \( e_i, e_j \in V \).

Now let \( t_{ij} = t_{ji} \).

Then \( f(e_i, e_j) = f(e_j, e_i) \) for all basis vectors \( e_i \).

But for all \( x, y \in V \) we have

\[
 f(x, y) = x^i y^j f(e_i, e_j) = x^i y^j f(e_j, e_i) = y^j x^i f(e_j, e_i) = y^j x^i f(e_i, e_j) = f(y, x)
\]