17. \( \mathbb{R} \) as a metric space

Consider the reals \( \mathbb{R} \) with \( \rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) defined by \( \rho(x, y) = |x - y| \). Show that this definition makes \( \mathbb{R} \) a metric space.

18. Limits of sequences

a) Show that a sequence in a metric space has at most one limit.

\textit{hint:} Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequence with a limit is a Cauchy sequence.

19. Banach space

Prove Proposition 1 from §4.6, i.e., show that the norm on the dual space \( B^* \) of a Banach space \( B \) as defined in §4.6 def. 4 is a norm in the sense of the norm \([-\ldots-]\) defined on \( B \) itself in §4.6 def. 1.

20. Hilbert space

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of §4.6 def. 1.

\textit{hint:} Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings \( \ell \) defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).
Positive definiteness of $xy$ can be shown.

From the triangle inequality:

By definition of $|x|$, we have $xy \leq |x||y| \quad \forall x, y \in \mathbb{R}$

$s \quad 0 \leq 2(x-y)(z-y) + 2|xy-zy|$  

$s \quad (x-z)^2 = x^2 - 2xz + z^2 \leq x^2 - 2xz + z^2 + 2(x-y)(z-y) + 2|z-y| \cdot |z-y|$  

$s \quad = x^2 - 2xz + z^2 + 2(x-y)(z-y) + 2|z-y| \cdot |z-y|$  

$s \quad = x^2 - (x+y)^2 + 2(x-y)(z-y) + 2|z-y| \cdot |z-y|$  

$s \quad = x^2 - (x-y)^2 + (y-z)^2 + 2|z-y| \cdot |z-y|$  

$s \quad = (x-z)^2 + (y-z)^2 + 2|z-y| \cdot |z-y|$  

$s \quad (x-z)^2 \geq 0 \Rightarrow$  

$s \quad |x-z| \leq |x-y| + |y-z| \quad \text{high inequality}$
12. a) Let \( x_n \to x \) in \( X \). Suppose \( x_n \to y \) in \( X \).

\[
\Rightarrow f(x^*, y^*) \leq f(x, x_n) + \varepsilon f(y, x_n) \quad \forall \varepsilon > 0 \text{ by the triangle inequality.}
\]

\[
\lim_{n \to \infty} f(x, x_n) = \lim_{n \to \infty} f(y, x_n) = 0
\]

\[
\Rightarrow f(x^*, y^*) = 0 \quad \Rightarrow x^* = y^*
\]

b) Let \( x_n \) have a limit \( x^* \): \( x_n \to x^* \)

\[
\Rightarrow f(x_n, x_n) \leq f(x_n, x^*) + f(x_n, x^*)
\]

Let \( \delta > 0 \). Then \( \exists N \in \mathbb{N} : f(x_n, x^*) < \delta \quad \forall n > N \)

Now let \( \varepsilon > 0 \) and \( \delta = \varepsilon/2 \). Then \( \exists N > 0 : \)

\[
f(x_n, x_n) \leq f(x_n, x^*) + f(x_n, x^*) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

provided \( n, m > N \).
10.3) And the solutions for \( \mathcal{L}(x) \) def. 1:

\[
\text{(i) } \|x\| = \inf \left\{ \|y\| \mid \|y - x\| = 0 \right\}
\]

The null vector in \( \mathcal{V} \) is the null function \( f \) defined by \( f(x) = 0 \) \( \forall x \in \mathcal{V} \).

\[
\Rightarrow \|f\| = 0
\]

Conversely, let \( \|x\| = 0 \). Then \( x \) is of \( \|x\| = 0 \) implies \( x = 0 \), \( x \) is unique to \( 0 \).

\[
\Rightarrow \|x\| = 0 \iff x = 0
\]

\[
\text{(ii) } \|x + y\| = \inf \left\{ \|v + w\| \mid \|v - x\| + \|w - y\| = 0 \right\}
\]

\[
\leq \inf \left\{ \|v + w\| : \|v - x\| + \|w - y\| = 0 \right\}
\]

\[
= \|x\| + \|y\|
\]

That is, \( \mathcal{N}(x) \) is the closed set for \( \mathcal{V} \).

\[
\text{(iii) } \|x\| = \inf \left\{ \|v\| : \|v - x\| = 0 \right\}
\]

\[
= \inf \left\{ \|v\| : \|v - x\| = 0 \right\}
\]

\[
\leq \inf \|v\| \forall v \in x
\]

\[
\Rightarrow \|x\| + c \in E, c \in \mathcal{V}
\]
20. (ii) \( \| x + y \|^2 = (x, x) + (y, y) + (x, y) + (y, x) \)
\[ = (x, x) + (y, y) + (x, y) + (y, x) \]
\[ = \| x \|^2 + \| y \|^2 + 2 \text{Re}(x, y) \]

\( \Rightarrow \) \( \text{Re}(x, y) = \left| (x, y) \right| - \left| (y, x) \right| \leq \left| (x, y) \right| \)
\[ \leq (x, x)(y, y) = \| x \|^2 \cdot \| y \|^2 \]
\[ \Rightarrow \| x + y \|^2 \leq \| x \|^2 + \| y \|^2 + 2 \| x \| \cdot \| y \| \]
\[ \Rightarrow \| x + y \| \leq \| x \| + \| y \| \]

(iii) \( \| ax \|^2 = (ax, ax) = a^2(x, x) = |a|^2 \cdot \| x \|^2 \)
\[ \Rightarrow \| ax \| = |a| \cdot \| x \| \quad \forall a \in \mathbb{C}, x \in \mathbb{H} \]

\( \Rightarrow \) \( \| \cdot \| \) is a norm

b) By definition \( \ell(x) = (y, x) \) is linear.

(i) \( \ell(x + t) = (y, x + t) = (y, x) + (y, t) = \ell(x) + \ell(t) \quad \forall x, t \in \mathbb{H} \)

(ii) \( \ell(\lambda x) = (y, \lambda x) = \lambda (y, x) = \lambda \ell(x) \quad \forall x \in \mathbb{H}, \lambda \in \mathbb{C} \)

\( \Rightarrow \) \( \ell \) is a linear form