30. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

\[(\kappa^2 - \nabla^2)\varphi(x) = 4\pi \rho(x)\]

in terms of an integral.

*hint:* The answer is a generalization of Poisson’s formula.

(3 points)

31. **Electric charges in an external field**

Consider a static electric charge distribution \(\rho(x)\) subject to a static potential \(\varphi(x)\). Consider the force \(F_{\text{el}}\) on the charge distribution and show that \(F_{\text{el}} = -\nabla U\), with \(U\) the electrostatic energy calculated in ch.3 §3.6. In particular, convince yourself that the dipole term in the multipole expansion of \(U\) gives the correct potential energy for an electric dipole moment \(d\) in an electric field \(E\).

(3 points)

32. **Electromagnetic waves and gauge invariance**

a) Show that the Lorenz gauge, \(\frac{1}{c} \partial_t \varphi + \nabla \cdot A = 0\), still does not uniquely determine the potentials of an electromagnetic wave: Let \(f\) be an arbitrary scalar solution of the wave equation, \(\Box f = 0\). Then the transformation \(A \rightarrow A + \nabla f\), \(\varphi \rightarrow \varphi - \frac{1}{c} \partial_t f\) leaves both the wave equation for the 4-vector potential and the fields unchanged.

b) Show in particular that the gauge of an electromagnetic wave can always be chosen such that \(\varphi = 0\), \(\nabla \cdot A = 0\).

(3 points)

33. **Plane waves**

Consider the scalar field

\[\psi(x, t) = \cos(k \cdot x - \omega t)\]

where \(k\) is a Euclidian vector.

a) What is necessary and sufficient to make \(\psi\) a solution of the wave equation?

b) Perform a Lorentz boost, and show that the transformed wave again has the form

\[\psi'(x', t') = \cos(k' \cdot x' - \omega' t')\]

How are \(k'\) and \(\omega'\) related to \(k\) and \(\omega\)?

(3 points)