38. **Polaritons**

As a model for a dielectric, consider a polarization field \( P(x,t) \) that determines the sources of the electromagnetic fields according to

\[
j = \partial_t P, \quad \rho = -\nabla \cdot P.
\]

In addition to Maxwell’s equations, the dynamics of the system are governed by an equation of motion for \( P \),

\[
(\partial_t^2 + \omega_0^2)P(x,t) = a^2 E(x,t) \quad (*),
\]

where \( \omega_0 \) is a characteristic frequency and \( a \) is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

a) Show that Maxwell’s equations plus (*) have solutions given by both longitudinal \((k \parallel E, P)\) and transverse \((k \perp E, P)\) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.

b) Show that the transverse waves in the long-wavelength limit are photon-like, viz., \( \omega_T(k \to 0) = \frac{c}{n}|k| \), and determine the index of refraction \( n \).

c) Show that no homogeneous wave propagation is possible in a frequency band \( \omega_- < \omega < \omega_+ \), and find \( \omega_\pm \).

Derive the Lyddane-Sachs-Teller relation

\[
\omega_+^2/\omega_-^2 = \epsilon(\omega = 0)
\]

where \( \epsilon(\omega) = 1 + 4\pi a^2/\omega_0^2 - \omega^2 \) is the dielectric function of the dielectric.

d) Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits \( k \to 0 \) and \( k \to \infty \), and plot the result.

(14 points)

39. **Liédard-Wiechert potentials**

Consider a point charge \( e \) that moves on a given trajectory \( X(t) \) with velocity \( v(t) = \dot{X}(t) \) which results in charge and current densities

\[
\rho(x,t) = e \delta(x - X(t)), \quad j(x,t) = e v(t) \delta(x - X(t))
\]

Show that the resulting retarded potentials have the form

\[
\varphi(x,t) = \frac{e}{|x - X(t_-)| - v(t_-) \cdot (x - X(t_-)) / c},
\]

\[
A(x,t) = \frac{1}{c} v(t_-) \varphi(x,t)
\]

where \( t_- \) is the solution of

\[
t_- = t - \frac{1}{c} |x - X(t_-)| \quad (*)
\]

.../over
These are known as Liénard-Wiechert potentials after Alfred-Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

hint: Show that the equation \((*)\) for \(t_-\) has one and only one solution. (6 points)

40. Potential of a uniformly moving charge

Consider a charge \(e\) moving uniformly along the \(x\)-axis with velocity \(v\): \(X(t) = (vt, 0, 0)\). Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 3 §2.4 by means of a Lorentz transformation. (6 points)