21. Planar charge distributions

a) Consider a homogeneously charged infinitesimally thin ring with radius $R$ and total charge $Q$ that is oriented perpendicular to the $z$-axis. Calculate the electric field on the $z$-axis.

b) The same for a homogeneously charged disk with charge density $\sigma$ and radius $R$. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

22. Spherically symmetric charge distributions

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(x) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

**hint:** Make an ansatz for a purely radial field, $E(x) = E(r) \mathbf{\hat{r}}$, and integrate Gauss’s law over a spherical volume.

Explicitly calculate and plot the field $E(x)$ and the potential $\varphi(x)$ for

b) a homogeneously charged sphere

\[ \rho(x) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases} \]

c) a homogeneously charged spherical shell

\[ \rho(x) = \sigma_0 \delta(r - r_0) \]

(8 points)

23. 2-d Levi-Civita tensor

Consider $\mathbb{R}_2$ as a Euclidian space with cartesian basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let $\epsilon(x, y)$ be the antisymmetric bilinear form defined by $\epsilon(e_1, e_2) = 1$. Let

\[ \epsilon_{ij} = \begin{cases} +1 & \text{if } i = 1, j = 2 \\ -1 & \text{if } i = 2, j = 1 \\ 0 & \text{otherwise} \end{cases} \]
a) Show that the most general antisymmetric bilinear form \( a(x, y) \) is given by

\[
a(x, y) = a(e_1, e_2) \epsilon(x, y)
\]

b) Show that \( \epsilon(x, y) \) is the determinant of the matrix \( \begin{pmatrix} x^1 & y^1 \\ x^2 & y^2 \end{pmatrix} \) whose columns are formed by the components of the vectors \( x \) and \( y \).

c) Show that for any antisymmetric bilinear form

\[
a(Dx, Dy) = (\det D) x(x, y)
\]

where \( D \) is an arbitrary coordinate transformation.

d) Show that the 2-d Levi-Civita tensor \( (\epsilon_L)_{ij} = \epsilon(e_i, e_j) \) transforms as a rank-2 tensor under orthogonal transformations.

e) Show that the 2-d Levi-Civita symbol, i.e. \( \epsilon_{ij} \) associated with all coordinate systems, transforms as a rank-2 pseudotensor, and that the one-component object

\[
y^i = \epsilon^{ij} x_j
\]

transforms as a pseudovector.

(8 points)