10. **Energy-momentum tensor**

Consider the electromagnetic field in the absence of matter.

a) Show that the tensor field

\[ H_{\mu}^{\nu}(x) = (\partial_{\mu}A_{\alpha}(x)) \frac{\partial \mathcal{L}}{\partial (\partial_{\nu}A_{\alpha}(x))} - \delta_{\mu}^{\nu} \mathcal{L} \]

obeys the continuity equation

\[ \partial_{\nu} H_{\mu}^{\nu}(x) = 0 \quad (\ast) \]

*note:* Notice that \( H_{\mu}^{\nu}(x) \) is a generalization of Jacobi’s integral in Classical Mechanics.

b) Show that (\ast) also holds for

\[ \tilde{T}_{\mu}^{\nu} = H_{\mu}^{\nu} + \partial_{\alpha} \psi_{\mu}^{\nu \alpha} \]

where \( \psi_{\mu}^{\nu \alpha}(x) \) is any tensor field that is antisymmetric in the second and third indices,

\[ \psi_{\mu}^{\nu \alpha}(x) = -\psi_{\mu}^{\alpha \nu}(x). \]

c) Show that \( \psi_{\mu}^{\nu \alpha} \) can be chosen such that \( \tilde{T}_{\mu}^{\nu}(x) = T_{\mu}^{\nu}(x) \), which provides an alternative proof that \( T_{\mu}^{\nu}(x) \) obeys (\ast).

(5 points)

11. **Energy-momentum conservation in the presence of matter**

Prove the corollary of ch. 1 §2.3: In the presence of matter, the energy-momentum tensor obeys the continuity equation

\[ \partial_{\nu} T_{\mu}^{\nu}(x) = -\frac{1}{c} F_{\mu}^{\nu}(x) J_{\nu}(x) \]

(2 points)

12. **Energy-momentum tensor for a massive scalar field**

Consider the massive scalar field from Problem 7:

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{m^{2}}{2} \phi^{2} \]

and the tensor field \( H_{\mu}^{\nu} \) defined analogously to Problem 13:

\[ H_{\mu}^{\nu} = (\partial_{\mu}\phi) \frac{\partial \mathcal{L}}{\partial (\partial_{\nu}\phi)} - \delta_{\mu}^{\nu} \mathcal{L} \]

Determine \( H_{\mu}^{\nu} \) explicitly and show that

\[ \partial_{\nu} H_{\mu}^{\nu} = 0 \]

*hint:* Use the Euler-Lagrange equation determined in Problem 7a).

(3 points)

.../over
13. **Coulomb gauge**

Consider the 4-vector potential \( A^\mu(x) = (\varphi(x), A(x)) \). Show that one can always find a gauge transformation such that

\[ \nabla \cdot A(x) = 0 \]

This choice is called *Coulomb gauge*. (2 points)
\[ 0 = \partial_\nu \left( \frac{\partial x}{\partial (\partial_\mu \lambda)} \partial_\lambda A_\nu - \delta_\nu^\mu \right) = \partial_\nu H_\nu^\mu \]

b) \[ \partial_\nu \partial_\lambda \chi_\mu = - \partial_\nu \partial_\lambda \chi_\mu = - \partial_\lambda \partial_\nu \chi_\mu = - \partial_\nu \partial_\lambda \chi_\mu \]

\[ \partial_\nu \partial_\lambda \chi_\mu = 0 \quad \Rightarrow \quad \partial_\nu H_\nu^\mu = 0 \]

c) \[ \chi_\mu = \frac{1}{\epsilon_{\mu}} A_\mu F^{\nu \mu} = \frac{1}{\epsilon_{\mu}} A_\mu F^{\nu \mu} \]

\[ \Rightarrow \partial_\nu H_\nu^\mu = 0 \quad \Rightarrow \quad \partial_\nu H_\nu^\mu = 0 \]

\[ \chi_\mu = \frac{1}{\epsilon_{\mu}} A_\mu F^{\nu \mu} = (\partial^\nu A_\mu) \frac{\partial x}{\partial (\partial^\nu A_\mu)} - \eta^{\nu \mu} \chi + \frac{i}{\epsilon} \partial_\lambda A_\mu \xi^{\lambda \nu} = 0 \]

\[ \Rightarrow \partial_\nu H_\nu^\mu = 0 \]

\[ \Rightarrow \partial_\nu H_\nu^\mu = 0 \]
11.) Generalize the proof of the proposition in § 2.2.

The only difference is that now the electric field

\[ \frac{\partial E^y}{\partial x} = \frac{\varepsilon}{x} \frac{\partial}{\partial x} \]

\[ \Rightarrow \frac{\partial F^y}{\partial x} = \frac{1}{\varepsilon \sigma} \left[ - (\partial_y F^x) F^y - F^y \partial_x F^y + \frac{\varepsilon}{\sigma} (\partial_x F^x) F^y \right] \]

\[- \frac{1}{\varepsilon} F^y d^x \frac{\partial}{\partial x} \]

\[- \frac{1}{\varepsilon} F^y d^y \]

\[ = 0 \quad \text{by } § 2.2 \]
\[ \mu_\nu = (\partial_\mu \phi) \frac{\partial \phi}{\partial (\partial_\nu \phi)} - \delta_\mu^\nu \frac{\partial \phi}{\partial \partial_\nu \phi} \]

\[ = (\partial_\mu \phi)(\partial_\nu \phi) - \delta_\mu^\nu \frac{\partial \phi}{\partial \partial_\nu \phi}(\partial_\nu \phi) + \delta_\mu^\nu \frac{\partial \phi}{\partial \partial_\nu \phi} \phi \]

\[ \Rightarrow \partial_\nu \mu_\mu = (\partial_\nu \partial_\mu \phi)(\partial_\nu \phi) + (\partial_\mu \phi)(\partial_\nu \partial_\nu \phi) - (\partial_\mu \phi)(\partial_\nu \partial_\nu \phi) + \phi \partial_\nu \phi \phi \]

\[ = (\partial_\nu \partial_\mu \phi)(\partial_\nu \phi) - (\partial_\mu \phi)(\partial_\nu \partial_\nu \phi) + (\partial_\mu \phi)(\partial_\nu \partial_\nu \phi) + \phi \partial_\nu \phi \phi \]

\[ = 0 \]

by Min. - Gordon Eq. by Min. - Gordon Eq. by Min. - Gordon Eq.
P.1

(1) Group loop: \[ A^T \rightarrow A^T - \Theta^T \chi \]

\[ \rightarrow \ddot{A} \rightarrow \ddot{A} - \ddot{\Theta} \chi \]

\[ \rightarrow \ddot{\nabla} \ddot{A} \rightarrow \ddot{\nabla} \ddot{A} - \ddot{\nabla}^2 \chi \]

Now down \( \chi \) as by virtue of the Poisson eq.

\[ \ddot{\nabla}^2 \chi (x) = \ddot{\nabla} \ddot{\Theta} (x) \]

The transformed \( \ddot{A} \) has the property

\[ \ddot{\nabla} \ddot{A} (x) = \ddot{\nabla} \ddot{A} (x) - \ddot{\nabla}^2 \chi (x) = 0 \]