21. **Planar charge distributions**

a) Consider a homogeneously charged infinitesimally thin ring with radius $R$ and total charge $Q$ that is oriented perpendicular to the $z$-axis. Calculate the electric field on the $z$-axis.

b) The same for a homogeneously charged disk with charge density $\sigma$ and radius $R$. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

22. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(x) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

*hint:* Make an *ansatz* for a purely radial field, $E(x) = E(r) \hat{e}_r$, and integrate Gauss’s law over a spherical volume.

Explicitly calculate and plot the field $E(x)$ and the potential $\varphi(x)$ for

b) a homogeneously charged sphere

$$\rho(x) = \begin{cases} \rho_0 & \text{if } r \leq r_0 \\ 0 & \text{if } r > r_0 \end{cases} .$$

c) a homogeneously charged spherical shell

$$\rho(x) = \sigma_0 \delta(r - r_0) .$$

(8 points)

23. **2-d Levi-Civita tensor**

Consider $\mathbb{R}_2$ as a Euclidian space with cartesian basis $e_1 = (1, 0)$, $e_2 = (0, 1)$. Let $\epsilon(x, y)$ be the antisymmetric bilinear form defined by $\epsilon(e_1, e_2) = 1$. Let

$$\epsilon_{ij} = \begin{cases} +1 & \text{if } i = 1, j = 2 \\ -1 & \text{if } i = 2, j = 1 \\ 0 & \text{otherwise} \end{cases} .$$
a) Show that the most general antisymmetric bilinear form \( a(x, y) \) is given by
\[
a(x, y) = a(e_1, e_2) \epsilon(x, y)
\]

b) Show that \( \epsilon(x, y) \) is the determinant of the matrix \( \begin{pmatrix} x^1 & y^1 \\ x^2 & y^2 \end{pmatrix} \) whose columns are formed by the components of the vectors \( x \) and \( y \).

c) Show that for any antisymmetric bilinear form
\[
a(Dx, Dy) = (\det D) x(x, y)
\]
where \( D \) is an arbitrary coordinate transformation.

d) Show that the 2-d Levi-Civita tensor \( (\epsilon_L)_{ij} = \epsilon(e_i, e_j) \) transforms as a rank-2 tensor under orthogonal transformations.

e) Show that the 2-d Levi-Civita symbol, i.e. \( \epsilon_{ij} \) associated with all coordinate systems, transforms as a rank-2 pseudotensor, and that the one-component object
\[
y^i = \epsilon^{ij} x_j
\]
transforms as a pseudovector.

(8 points)
21. (a) Let the wire be in the \( t=0 \) plane:
\[ s(t) = s_0 \delta(t) \delta(r-R) \]

in cylindrical coordinates.

Total charge:
\[ \int \delta(r-R) = 2\pi s_0 = q \]

Potential function:
\[ \phi(x) = \int \frac{\delta(r-R)}{|x-x'|} \]

Electric field:
\[ E = -\nabla \phi = -\int \frac{\delta(r-R)}{|x-x'|} \frac{1}{|x-x'|} = \int \delta(r-R) \frac{x-x'}{|x-x'|^3} \]

Symmetry -> \[ E(x=0,0,t) = E(t) \hat{\mathbf{t}} \]
\[ E(t) = \int \frac{s(t)}{|x-x'|^3} = \int \frac{2\pi s_0}{(t^2+r^2)^{3/2}} = \frac{q t}{(t^2+r^2)^{3/2}} \]

(b) Charge density:
\[ \frac{\partial \rho}{\partial t} \]

\[ \rho \text{ over a small interval } r \text{, with mass } dr: \]
\[ dQ = \frac{\rho}{2\pi r} 4\pi r dr = \frac{\rho}{2\pi} r dr = \frac{dQ}{dr} \quad Q = \frac{4}{3} \pi \rho R^3 \]

\[ E(t) = \int \frac{\rho}{(2\pi R^2)^{3/2}} \left[ \frac{r}{(t^2+r^2)^{3/2}} \right] = \frac{q t}{R^2} \frac{1}{t}\int \frac{dx}{(x^2+t^2)^{3/2}} \]
\[ = \frac{q t}{R^2} \left[ -\frac{1}{(4t^2)^{1/2}} \right] = \frac{q t}{R^2} \left[ \left(1 - \frac{t}{R^2} \right) + O(1) \right] \]
\[ E(t=\infty) = \frac{q t}{R^2} \left(1 - \frac{t}{R^2} + O(1) \right) = q t + O(t^{-1}) \quad \text{field of point charge} \]
\[ E(\infty_0) = E(\infty_\infty) = \infty \]

An infinite sheet with water being drained produces a field that's independent of \( \infty \).
22. 2) Wirds Gauss's law \( \nabla \cdot E = \frac{4\pi}{\varepsilon_0} \)

\[ \int \nabla \cdot E \, dV = \int \nabla \cdot E \, dV = 4\pi \int \nabla \cdot E \, dV \]

\( (v) \)

let \( \mathcal{S}(x) \) be spherically symmetric, \( \mathcal{S}(x) = \mathcal{S}(r) \), and note a

so that:

\( E(x) = E(r) \hat{r} \)

\( \hat{r} = \frac{x}{|x|} \)

\[ E(r) = \frac{4\pi}{r^2} \int dV \mathcal{S}(r') \]

\( r' \)

\( \mathcal{S}(r') \)

For the point charge, wirds \( E(x) = \nabla \varphi(x) \)

spherically symmetric \( \Rightarrow \nabla \varphi = \partial_r \hat{r} \varphi \)

\( \Rightarrow E(r) = -\partial_r \varphi(r) \)

\( \Rightarrow \varphi(r) = \int_0^r dr' E(r') \)

if \( \varphi \) is given \( \varphi(r=\infty) = 0 \)

\( b) \quad E(r) = \frac{4\pi}{r^2} \int_0^r dV \mathcal{S}_0 \theta(r'<r_0) \)

1st case: \( r < r_0 \)

\[ E(r) = \frac{4\pi}{r^2} \int_0^r \frac{dV}{s_0} = \frac{4\pi}{r^2} \frac{1}{3} r^3 = \frac{4\pi}{3} s_0 r \]

\[ \Rightarrow \frac{\frac{4\pi}{r^2} \frac{r}{r_0^2}}{\frac{1}{3} r_0^2} = \frac{A r}{r_0^2} \]

will \( A = \frac{4\pi}{3} \frac{r}{r_0^2} \)
\[ E(r) = \frac{Qr}{r^2} \] for \( r < r_0 \)

\[ E(r) = \frac{Q}{r^2} \] for \( r > r_0 \)

\[ \vec{E}(\vec{r}) = \vec{E}(r) \hat{r} \]

For \( r < r_0 \):
\[ \phi(r) = \frac{Q}{r} \]
\[ \phi(r) = \left\{ \begin{array}{ll} \frac{Q}{2r_0^3}(2r_0^2 - r^2) & \text{for } r < r_0 \\ 0 & \text{for } r > r_0 \end{array} \right. \]

For \( r > r_0 \):
\[ \phi(r) = \int_0^{r_0} \frac{Qr'}{r_0^3} dr' + \int_{r_0}^{\infty} \frac{Qr'}{r'^3} dr' = \frac{Q}{2r_0^3} \left( r_0^3 - r^3 \right) + \frac{Q}{r_0} \]

For \( r < r_0 \), \( \hat{E} = \hat{r} \frac{Q}{r} \)

For \( r > r_0 \), \( \hat{E} = \hat{r} \frac{Q}{r_0^3} \left( 2r_0^2 - r^2 \right) \)

Electric field:
\[ E(r) = \left\{ \begin{array}{ll} 0 & \text{for } r < r_0 \\ \frac{Q}{r_0^3} & \text{for } r > r_0 \end{array} \right. \]

\[ E(r) = \frac{Q}{r_0^3} \left( 2r_0^2 - r^2 \right) \] for \( r > r_0 \)

\[ \frac{Q}{r_0^3} \]

\[ \frac{Q}{r_0^3} \]

For \( r > r_0 \), \( E(r) \) is the same as for the homogeneous sphere.

\[ Q = 4\pi r_0^2 \sigma_0 \text{ is total charge} \]
\[ p(r) = 0 \]

\[ \rho(r) : \begin{cases} \rho < \rho_0 & \text{for } r < r_0 \\ \rho = \frac{Q}{r^2} & \text{for } r \geq r_0 \end{cases} \]

\[ \int_{r_0}^{\infty} \frac{Q}{r^2} \, dr = Q r_0 \]

\[ \int_{r}^{\infty} \frac{Q}{r^2} \, dr = \frac{Q}{r} \]

\[ \phi(r) = \begin{cases} Q r_0 & \text{for } r < r_0 \\ \frac{Q}{r} & \text{for } r \geq r_0 \end{cases} \]
Problem 23: \( \mathbb{R}^d \) inner-product space

Let \( \mathbb{R}^d \) be an inner product space and a symmetric bilinear form \( a(x,y) = a(y,x) \)

- \( a(x,y) = a(y,x) \)
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- \( a(x,y) = a(y,x) \)

Let \( \mathbb{R}^d \) be an inner product space and a symmetric bilinear form \( a(x,y) \), \( \forall x,y \in \mathbb{R}^d \)

Unproperty: \( a(\mathbf{e}_1, \mathbf{e}_2) = 1 \) \( \mathbf{e}_1, \mathbf{e}_2 \) are basis vectors

Let \( \mathbf{e}_1, \mathbf{e}_2 \) be basis vectors

(a) Show that the most general bilinear form is \( a(x,y) = a(\mathbf{e}_1, \mathbf{e}_2) x \cdot y \)

\[
\begin{align*}
a(x,y) &= a(\mathbf{e}_1, \mathbf{e}_2) x \cdot y \\
&= a(\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} \\
&= a(\mathbf{e}_1, \mathbf{e}_2) x^T y
\end{align*}
\]

where:

\[
\begin{align*}
a_{ij} &= a(\mathbf{e}_i, \mathbf{e}_j) \\
a_{ij} &= a(\mathbf{e}_j, \mathbf{e}_i) \\
a_{ij} &= a(\mathbf{e}_i, \mathbf{e}_j) \\
a_{ij} &= a(\mathbf{e}_j, \mathbf{e}_i)
\end{align*}
\]

\( a(\mathbf{e}_1, \mathbf{e}_1) = a(\mathbf{e}_2, \mathbf{e}_2) = \ldots = a(\mathbf{e}_d, \mathbf{e}_d) \)

\( a(\mathbf{e}_1, \mathbf{e}_2) = a(\mathbf{e}_2, \mathbf{e}_1) = \ldots = a(\mathbf{e}_d, \mathbf{e}_{d-1}) \)

\( a(\mathbf{e}_i, \mathbf{e}_j) = a(\mathbf{e}_j, \mathbf{e}_i) = 0 \) for \( i \neq j \)

(b) Show that \( a(x,y) \) is the determinant of the matrix \( \begin{pmatrix} x^T \\ y^T \end{pmatrix} \), whose columns are formed by the products of \( \mathbf{e}_i \) and \( \mathbf{e}_j \).

\[
\begin{align*}
a(x,y) &= a(\mathbf{e}_1, \mathbf{e}_2) x \cdot y \\
&= a(\mathbf{e}_1, \mathbf{e}_2) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} \\
&= a(\mathbf{e}_1, \mathbf{e}_2) x^T y
\end{align*}
\]

(c) Show that for any symmetric bilinear form \( a(x,y) \),

\[
\begin{align*}
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y})
\end{align*}
\]

solution:

\[
\begin{align*}
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y}) \\
a(\mathbf{x}, \mathbf{y}) &= a(\mathbf{x}, \mathbf{y})
\end{align*}
\]

where:

\[
\begin{align*}
a_{ij} &= a(\mathbf{e}_i, \mathbf{e}_j) \\
a_{ij} &= a(\mathbf{e}_j, \mathbf{e}_i) \\
a_{ij} &= a(\mathbf{e}_i, \mathbf{e}_j) \\
a_{ij} &= a(\mathbf{e}_j, \mathbf{e}_i)
\end{align*}
\]

\( a(\mathbf{e}_1, \mathbf{e}_1) = a(\mathbf{e}_2, \mathbf{e}_2) = \ldots = a(\mathbf{e}_d, \mathbf{e}_d) \)

\( a(\mathbf{e}_1, \mathbf{e}_2) = a(\mathbf{e}_2, \mathbf{e}_1) = \ldots = a(\mathbf{e}_d, \mathbf{e}_{d-1}) \)

\( a(\mathbf{e}_i, \mathbf{e}_j) = a(\mathbf{e}_j, \mathbf{e}_i) = 0 \) for \( i \neq j \)
\[
\det A^T = \det A^2 = (\det A)^2 \epsilon(x, y) = \det D \epsilon(x, y)
\]

\[\epsilon) \rightarrow \epsilon (dx, dy) = \epsilon (dz, dz) \epsilon (dx, dy) = \det D \epsilon(x, y) \]

\[d) \text{ Show that the 1-d Levi-Civita basis } (\epsilon_1)_j = \epsilon(e_i, e_j) \text{ transforms as a real-1 tensor when orthonormal basis.} \quad \text{where: } (\tilde{\epsilon_1})_j = \epsilon (dx, d\tilde{y}); \quad \det D \epsilon(e_i, e_j) = \det D \tilde{\epsilon}\]

\[\epsilon \tilde{D} \tilde{\epsilon} \tilde{d} \epsilon_j = \epsilon \tilde{D} \tilde{\epsilon} \tilde{d} e_j = \epsilon \tilde{D} \tilde{\epsilon} \tilde{D} e_d = \epsilon \tilde{D} \epsilon \tilde{d} \epsilon_1 = \epsilon \tilde{D} \epsilon \tilde{d} \epsilon_1 \]

\[d) \text{ Show that the 2-1 Levi-Civita symbol, i.e., } \epsilon_{ij} \text{ associated with } \epsilon \text{ coordinate transformation as a real-2 object,} \quad \text{where: } \tilde{\epsilon}_{ij} = \epsilon(e, e^j) \text{ as a hypervector.} \]

\[\text{where: } \tilde{\epsilon}_{ij} = \epsilon_{ij} \quad \epsilon \tilde{D} \tilde{\epsilon} \tilde{D} \epsilon_j = \epsilon \tilde{D} \tilde{\epsilon} \tilde{D} e^j = \epsilon \tilde{D} \epsilon \tilde{d} \tilde{\epsilon}_i \]

\[\epsilon \tilde{D} \tilde{d} \epsilon_j = \epsilon \tilde{D} \epsilon \tilde{d} \tilde{\epsilon}_j = \epsilon \tilde{d} \epsilon (\tilde{\epsilon}^j) \tilde{\epsilon} \quad \epsilon \tilde{\epsilon} = \epsilon \tilde{d} \tilde{\epsilon} \tilde{d} \epsilon_1 \]

\[\epsilon \tilde{d} \tilde{\epsilon} \tilde{D} \tilde{\epsilon} \tilde{D} \epsilon_1 = \epsilon \tilde{d} \tilde{\epsilon} \tilde{D} \tilde{\epsilon} \tilde{D} e_1 \]

\[\epsilon \tilde{d} \tilde{\epsilon} \tilde{D} \tilde{\epsilon} \tilde{D} \epsilon_1 = \epsilon \tilde{d} \tilde{\epsilon} \tilde{D} \tilde{\epsilon} \tilde{D} e_1 \]