PHYS 610 Mathematical Methods for Scientists F 2018

Midterm Exam 10/17/2018

This is a take-home exam that runs parallel with the homework. It is due in class on Wednesday, 10/31/2018. You can consult any inanimate resource you like, but please don't get help from live resources (no, not even your dog)!

If you encounter logical gaps in your proofs that you can't fill, state clearly what you are assuming to be true but could not prove, and continue.

Credit breakdown: 2 points for the Lemma, 4 points for that Proposition, 9 points for the Theorem, for a total of 15 points.

Prove the following

Theorem: Let X and Y be sets, and let $f : X \to Y$ be a bijective mapping. Then $\exists!\ f^{-1} : Y \to X$ called the *inverse* of f such that

$$f \circ f^{-1} = \mathrm{id}_Y \qquad \text{and} \qquad f^{-1} \circ f = \mathrm{id}_X \ .$$

f^{-1} is also bijective, and its inverse is $(f^{-1})^{-1} = f$.

Hint: It is useful to first prove the following

Proposition: Let $f : X \to Y$ be surjective. Then $\exists\ g : Y \to X$ such that g is injective and $f \circ g = \mathrm{id}_Y$.

For this, in turn, it is useful to first prove the

Lemma: Let $f : X \to Y$, $g : Y \to X$, and $f \circ g = \mathrm{id}_Y$. Then f is surjective and g is injective.

Note: There is a point in the proof of the proposition where you need to make use, consciously or otherwise, of the

Axiom of Choice: Let I be an index set and $X_i \neq \varnothing \quad \forall i \in I$. Then the cartesian product $\prod_{i \in I} X_i \neq \varnothing$. Or, equivalently and in plain English: Given a class of nonempty sets there exists a "choice function" that picks from each set one of its elements. (This sounds trivial, but logically it is not, and the realization that it isn't has a very interesting history.)

To apply this in the current context, consider Y the index set.