PHYS 622 Classical Electrodynamics W 2019

Midterm Exam 02/06/2019
due 02/20/2019

Instructions: Treat this problem exactly as if it were a closed-book in-class exam, except that you can work on it during any contiguous 110-minute time slot of your choosing. That is, once you open the file the 110-minute clock starts ticking; you can't look at it, think about it, and then work on it later. And closed-book of course means no notes, books, computers, etc. It will be available starting on 2/6 after class, and it will be due on 2/20 in class.

Charged Klein-Gordon field **(22 pts)**

Consider a Lagrangian density for a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L}_0 = (\partial_\mu \phi(x)) (\partial^\mu \phi(x))^* - m^2 |\phi(x)|^2$$

where x is a point in Minkowski space and ∂_μ is the 4-gradient.

a) Give a *simple* argument (**no** calculations!) that shows that $\mathcal{L}_0$ is invariant under Lorentz transformations. (1pt)

b) Show that $\mathcal{L}_0$ is invariant under *global* gauge transformations $\phi(x) \to e^{i\Lambda}\phi(x)$, with Λ independent of x. (1pt)

c) Show that $\mathcal{L}_0$ is *not* invariant under *local* gauge transformations $\phi(x) \to e^{i\lambda(x)}\phi(x)$, where λ is an arbitrary real function of x. (2 pts)

Now consider the modified Lagrangian density

$$\mathcal{L} = (D_\mu \phi(x)) (D^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x)$$

with $D_\mu = \partial_\mu - iqA_\mu$, where A_μ is the 4-vector potential and $F_{\mu\nu}$ is the electromagnetic field tensor.

d) Show by the same simple argument as in part a) that $\mathcal{L}$ is invariant under Lorentz transformations, and by an explicit calculation that in addition it is invariant under local gauge transformations $\phi(x) \to e^{iq\lambda(x)}\phi(x)$, $A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x)$. (7 pts)

e) Derive the Euler-Lagrange equations for the action

$$S = \int d^4x\, \mathcal{L}$$

hint: You may use, without derivation, $\partial\mathcal{L}/\partial(\partial_\mu A_\nu) = F^{\nu\mu}/4\pi$, which we derived in class.

Show that two of the Euler-Lagrange equations are generalizations of the Klein-Gordon equation $(\partial_\mu \partial^\mu + m^2)\phi = 0$, while the third one is the electromagnetic field equation with a particular 4-current density (which you may recognize from Quantum Mechanics). (11pts)