Chapter 2 Topics in Analysis

§1 Reminder: Real Analysis (see Math 251-3 + 281,2 or equivalent)

1.1 Differentiation and integration

Consider mappings ("functions") $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$. We say that $\vec{f}$ is a (m-vector-valued) function of n real variables and write

$$\vec{f}(\vec{x}) \equiv f(x_1, \dots, x_n) = \vec{y} \quad , \quad \vec{x} = (x^1, \dots, x^n) \in \mathbb{R}^n$$
$$\vec{y} = (y^1, \dots, y^m) \in \mathbb{R}^m$$

For $m = 1$ we write f instead of $\vec{f}$.

def. 1: (a) For $n = m = 1$ we define the derivative of f, $f' \equiv \frac{df}{dx} : \mathbb{R} \to \mathbb{R}$ by

$$\boxed{f'(x) := \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x+\varepsilon) - f(x) \right]} \quad (*)$$

and higher derivatives by $d^2f/dx^2 := \frac{d}{dx}\frac{df}{dx}$ etc.

(b) For $n > 1$, $m = 1$ we define partial derivatives $\frac{\partial f}{\partial x^i} \equiv \partial_i f : \mathbb{R}^n \to \mathbb{R}$

by (*) applied to the argument x^i, and the gradient of f

$\frac{\partial f}{\partial \vec{x}} \equiv \vec{\nabla} f : \mathbb{R}^n \to \mathbb{R}^n$ by $\boxed{\vec{\nabla} f(\vec{x}) := (\partial_1 f(\vec{x}), \dots, \partial_n f(\vec{x}))}$

(c) For $n = 1$, $m > 1$ we define $d\vec{f}/dx : \mathbb{R} \to \mathbb{R}^m$ by $\frac{d\vec{f}}{dx} := \left(\frac{df_1}{dx}, \dots, \frac{df_m}{dx}\right)$

(d) For $n = m$ we define the divergence $\operatorname{div} \vec{f} \equiv \vec{\nabla} \cdot \vec{f} : \mathbb{R}^n \to \mathbb{R}$

by $\vec{\nabla} \cdot \vec{f}(\vec{x}) = \partial_i f^i(\vec{x})$

(e) For $n = m = 3$ we define the curl $\operatorname{curl} \vec{f} \equiv \vec{\nabla} \times \vec{f} : \mathbb{R}^3 \to \mathbb{R}^3$

by $\boxed{\left(\vec{\nabla} \times \vec{f}(\vec{x})\right)^i := \varepsilon^{ij}{}_k \, \partial_j f^k(\vec{x})}$

remark: (1) If the space is Euclidean, then $\partial_i = \partial^i$, $\varepsilon^{ij}{}_k = \varepsilon_{ijk}$, etc.

def. 2: Let $I = [t_1, t_2] \subset \mathbb{R}$ and $\vec{x}: I \to \mathbb{R}^n$ a function of t.

Let $f: \mathbb{R}^n \times I \to \mathbb{R}$ be a real-valued fct. of $\vec{x}$ and t.

Then we define the total derivative of f with respect to t,

$df/dt: I \to \mathbb{R}$ by

$$\boxed{\frac{df}{dt}(t^*) := \partial_t f(\vec{x}(t^*), t^*) + \partial_i f(\vec{x}(t^*), t^*)\, \frac{dx^i}{dt}(t^*)}$$

proposition: Taylor expansion

Let $f: \mathbb{R}^n \to \mathbb{R}$ be m times diff'able at $\vec{x}$. Then there exists a neighborhood of $\vec{x}$ where f can be expanded by a power series

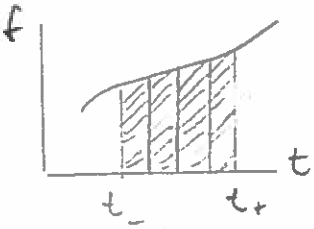
$$\boxed{\begin{aligned} f(x_1+\varepsilon, x_2, \dots, x_n) &= f(x_1, \dots, x_n) + \varepsilon \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) + \dots \\ &\quad + \frac{1}{m!}\varepsilon^m \frac{\partial^m f}{\partial x_1^m}(x_1, \dots, x_n) + r_m \end{aligned}}$$

and analogously for the other variables.

proof: Analysis course.

remark: (2) Taylor's theorem gives an explicit upper bound for the remainder r_m.

def. 3: Let $f: I \to \mathbb{R}$ be a real-valued fct of $t \in I = [t_-, t_+] \subset \mathbb{R}$. Then the Riemannian integral



$$\boxed{F = \int_{t_-}^{t_+} dt\, f(t) := \lim_{N \to \infty} \sum_{i=1}^{N-1} f(t_i)(t_{i+1} - t_i)} \qquad t_1 = t_-, \quad t_N = t_+$$

is defined as the limit of a sum, provided the limit exists.

remark: (3) Generalization to $f: I_1 \times I_2 \to \mathbb{R}$, $F = \int_{t_-}^{t_+} dt \int_{u_-}^{u_+} du\, f(t,u)$ is straightforward.

(4) F is a special case of a functional, i.e., a mapping

[illegible]

1.2 Paths, and line integrals

def. 1: (a) let $\mathcal{J} = [t_-, t_+] \subset \mathbb{R}$ and let $\vec{q} : \mathcal{J} \to \mathbb{R}^n$ be cont. diff'ble.

Then the set $\mathcal{C} := \{\vec{q}(t), t \in \mathcal{J}\} \subset \mathbb{R}^n$ is called a path or curve in $\mathbb{R}^n$, and $\vec{q}(t)$ is called a parametrization of $\mathcal{C}$ with parameter t.

(b) $\mathcal{C}$ inherits an order from the $\geq$ order defined on $\mathcal{J}$:

$$\vec{q}(t_1) < \vec{q}(t_2) \text{ by definition iff } t_1 < t_2$$

(c) The tangent vector $\vec{\tau}(t)$ in the point $\vec{q}(t)$ is defined as

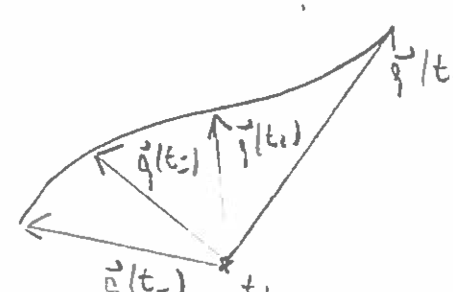
$$\boxed{\vec{\tau}(t) := \frac{d}{dt}\vec{q}(t) \equiv \dot{\vec{q}}(t)}$$

def. 2: let $L : \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{J} \to \mathbb{R}$ be a fct. of $\vec{q}, \dot{\vec{q}}$, and t that is sufficiently well behaved with respect to all arguments. Let $\vec{q}(t)$ be a parametrization of a path $\mathcal{C}$ and define a functional

$$\boxed{S_L(\mathcal{C}) := \int_{t_-}^{t_+} dt \, L(\vec{q}(t), \dot{\vec{q}}(t), t)}$$

Then for given L, $S_L(\mathcal{C})$ is characteristic of $\mathcal{C}$.

example: (1) $n = 2$. length of path $\mathcal{C}$ with parametrization $\vec{q}(t)$:



$$\underline{l_{\mathcal{C}}} = \lim_{N\to\infty} \sum_{i=1}^{N-1} \sqrt{(\vec{q}(t_{i+1}) - \vec{q}(t_i))^2}$$

$$= \lim_{N\to\infty} \sum_{i=1}^{N-1} (t_{i+1} - t_i) \sqrt{(\vec{q}(t_{i+1}) - \vec{q}(t_i))^2 / (t_{i+1} - t_i)^2} \underset{\S 1.1 \text{ def. } 3}{=} \int_{t_-}^{t_+} dt \sqrt{\vec{\tau}^2(t)}$$

$\to$ The choice

$$L(\vec{q}, \dot{\vec{q}}, t) = \sqrt{\dot{\vec{q}}^2} \text{ yields } S_L(\mathcal{C}) = l_{\mathcal{C}}.$$

(2) With L the Lagrangian of a mechanical system, S_L is the action (see PHYS 611).

def. 3: Let $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ be a fct. and $\mathcal{C}$ a path in $\mathbb{R}^n$ with parametrization $\vec{q}(t)$. Then the line integral of $\vec{F}$ over $\mathcal{C}$ is defined as

$$\boxed{\int_{\mathcal{C}} d\vec{l} \cdot \vec{F} := \int_{t_-}^{t_+} dt\, \vec{\tau}(t) \cdot \vec{F}(\vec{q}(t))}$$

with $d\vec{l} := \vec{\tau}(t)\, dt$ the integration measure.

remark: (1) Integration over a closed curve is denoted by $\oint_{\mathcal{C}} d\vec{l}$

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1.3 Surfaces, and surface integrals

def. 1: (a) Let $I_t = [t_-, t_+] \subset \mathbb{R}$ and $I_u = [u_-, u_+] \subset \mathbb{R}$ be intervals and let $\vec{r}: I_t \times I_u \to \mathbb{R}^3$ be a cont. differentiable fct. of t and u. Then

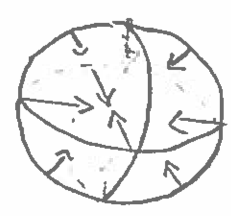
$$\boxed{S = \{ \vec{r}(t,u) \, ; \, (t,u) \in I_t \times I_u \}}$$

is called a surface in $\mathbb{R}^3$ with parametrization $\vec{r}(t,u)$

(b) The standard normal vector $\vec{n}(t,u)$ of S in $\vec{r}(t,u)$ is defined as

$$\boxed{\vec{n}(t,u) := \partial_t \vec{r}(t,u) \times \partial_u \vec{r}(t,u)}$$

example: (1) $I_\varphi = [0, 2\pi]$, $I_\vartheta = [0, \pi]$



$\vec{r}(\varphi, \vartheta) = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$

parametrizes a spherical surface in $\mathbb{R}^3$

The standard normal vector is

$\vec{n}(\varphi, \vartheta) = (-\sin^2\vartheta \cos\varphi, -\sin^2\vartheta \sin\varphi, -\sin\vartheta \cos\vartheta)$

def. 2: (a) Let $\vec f: \mathbb{R}^3 \to \mathbb{R}^3$ be a fct. and S a surface in $\mathbb{R}^3$ with parametrization $\vec r(t,u)$ and standard normal vector $\vec n(t,u)$. Then the surface integral of $\vec f$ over S is defined as

$$\boxed{\int_S d\vec\sigma\cdot\vec f := \int_{t_-}^{t_+} dt \int_{u_-}^{u_+} du\ \vec n(t,u)\cdot\vec f(\vec r(t,u))}$$

with $d\vec\sigma := \vec n(t,u)\,dt\,du$ the integration measure.

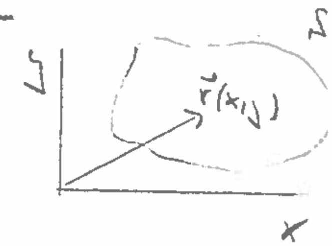
(b) The area of S is defined as

$$\boxed{A(S) := \int_{t_-}^{t_+} dt \int_{u_-}^{u_+} du\ |\vec n(t,u)|}$$

example: (2) surface area of the sphere from example 1: $|\vec n(\vartheta,\varphi)| = |\sin\vartheta|$

$$\to \underline{\underline{A}} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\ |\sin\vartheta| = 2\pi \int_{-1}^{1} d\cos\vartheta = \underline{\underline{4\pi}}$$

(3) A flat surface parametrized by the cartesian coordinates of its points: $\vec r(x,y,t) = (x,y,0)$



$$\to \vec n(x,y) = \begin{pmatrix}1\\0\\0\end{pmatrix}\times\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad \to \quad \underline{\underline{A(S) = \int_S dx\,dy}}$$

theorem 1: (Gauss) Let $V \subset \mathbb{R}^3$ be a volume with surface (V) and let $\vec f: \mathbb{R}^3 \to \mathbb{R}^3$ be a fct. Then

$$\boxed{\int_V dV\ \vec\nabla\cdot\vec f = \int_{(V)} d\vec\sigma\cdot\vec f}$$

with $dV = dx\,dy\,dz$ the measure of the volume integral

theorem 2: (Stokes) Let S be a surface in $\mathbb{R}^3$ bounded by a curve (S). Then

$$\boxed{\int_S d\vec\sigma\,(\vec\nabla\times\vec f) = \oint_{(S)} d\vec\ell\cdot\vec f}$$

Week 7

H7 (21, 22, 23, 24, 25)

proofs: Stoll 281,2 or equivalent