Problem Assignments # 9 11/21/2018
due 11/28/2018

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31. **Applications of the residue theorem**

Use complex analysis to evaluate the real integrals

a)

$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x}$$

hint: Write $\sin x = (e^{ix} - e^{-ix})/2i$ and consider the resulting two integrals with complex integrands. Why is this a good strategy?

b)

$$\int_{-\infty}^{\infty} dx\, \frac{\sin x}{x}\, \frac{1}{1+x^2}$$

and check your results by means of Wolfram Alpha.

Let $a \in \mathbb{C}$ with $\mathrm{Re}\, a > 0$. Use the residue theorem to show that

c)

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\pi/a}$$

Now let $a \in \mathbb{R}$ and consider the integral

d)

$$\int_{-\infty}^{\infty} \frac{dx}{x}\, \frac{1}{x^2+a^2}$$

and define its Cauchy principal value by

$$\lim_{R\to 0}\Big[\int_{-\infty}^{-R} dx\, f(x) + \int_{R}^{\infty} dx\, f(x)\Big]$$

with $f(x) = 1/x(x^2+a^2)$. Determine the Cauchy principal value using the residue theorem. Is the result consistent with the expectation for a real symmetric integral over an antisymmetric integrand?

hint: Go around the pole on a semicircle of radius R and let $R \to 0$.

(14 points)

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34. **More 1-d Fourier transforms**

Consider a function of time $f(t)$ and define its Fourier transform

$$\hat{f}(\omega) := \int dt\, e^{i\omega t}\, f(t)$$

and its Laplace transform $F(z)$ as

$$F(z) = \pm i \int dt\, e^{izt}\, f_\pm(t) \qquad (\pm \text{ for } \mathrm{sgn}(\mathrm{Im}\, z) = \pm 1)$$

with z a complex frequency and $f_\pm(t) = \Theta(\pm t)\, f(t)$. Further define

$$F''(\omega) = \frac{1}{2i}\left[F(\omega + i0) - F(\omega - i0)\right] \qquad , \qquad F'(\omega) = \frac{1}{2}\left[F(\omega + i0) + F(\omega - i0)\right]$$

Calculate $F''(\omega)$ and $F'(\omega)$ for

a) $f(t) = e^{-|t|/\tau}$

b) $f(t) = e^{i\omega_0 t}$

hint: $\lim_{\epsilon\to 0} \epsilon/(x^2 + \epsilon^2) = \pi\, \delta(x)$, with $\delta(x)$ the familiar Dirac delta-function, which we will study in detail in Week 10.

Show that in both cases $\int \frac{d\omega}{\pi} \frac{F''(\omega)}{\omega} = F'(\omega = 0)$.

note: These concepts are important for the theory of response functions.

(4 points)

35. **Regularizations of the constant function, and of the sign function**

Prove the following statements from PHYS 610 §3.4:

a) The sequence $f_n(x) = e^{-x^2/n^2}$ is a regular sequence of test functions that is a regularization of the generalized function $f(x) \equiv 1$.

b) The sequence $f_n(x) = \tanh(nx)$ is a regularization of the generalized function $f(x) = \mathrm{sgn}\, x$.

(4 points)

36. **Distribution limits**

a) Show that the sequences

$$f_n(x) = \frac{1}{\pi x} \sin(nx) \qquad (n = 1, 2, \ldots)$$

and

$$g_n(x) = \frac{1}{\pi n} \frac{1}{x^2 + 1/n^2} \qquad (n = 1, 2, \ldots)$$

yield the δ-function as $n \to \infty$ in a distribution-limit sense:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = \delta(x)$$

b) Show that

$$\frac{d}{dx}\,\mathrm{sgn}\,x = 2\,\delta(x)$$

and

$$\frac{d}{dx}\,\Theta(x) = \delta(x)$$

with $\Theta(x)$ the step function.

c) Show that

$$\lim_{\epsilon\to 0} \epsilon\,|x|^{\epsilon-1} = 2\,\delta(x)$$

hint: Start with ch.3 §2.4 example 6 and differentiate.

(6 points)