PHYS 622 | Classical Electrodynamics | W 2019

Problem Assignment # 1 | 01/09/2019
due 01/16/2019

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1. **Dual field tensor**

Show that the dual field tensor $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$ obeys

$$\partial_\mu \tilde{F}^{\mu\nu}(x) = 0$$

(2 points)

2. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon} \Big(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)] \Big)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

i) $F = \int dx\, \varphi(x)$

ii) $F = \int dx\, \varphi^2(x)$

iii) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
hint: Integrate by parts and assume that the boundary terms vanish.

iv) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is problematic. It does work for operational purposes, though.

b) Consider a “Lagrangian” $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ (i.e., a function of φ and its derivatives) and an “action” $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

3. **Massive scalar field**

a) Consider a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\varphi(x))\,(\partial^\mu\varphi(x)) - \frac{m^2}{2}\,(\varphi(x))^2$$

for a real scalar field $\varphi(x)$. Find the Euler-Lagrange equation for the field φ by requiring $\delta S/\delta\phi(x) = 0$.

b) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\,(\partial_\mu\phi(x))\,(\partial^\mu\phi^*(x)) - \frac{m^2}{2}\,|\phi(x)|^2$$

What are the Euler-Lagrange equations now?

c) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(3 points)

.../over

4. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\rm GL} = \int d\boldsymbol{x} \Big[r \left|\phi(\boldsymbol{x})\right|^2 + c \left| \big[\nabla - iq\boldsymbol{A}(\boldsymbol{x})\big] \phi(\boldsymbol{x}) \right|^2 + u \left|\phi(\boldsymbol{x})\right|^4 + \frac{1}{16\pi\mu} F_{ij}(\boldsymbol{x}) \, F^{ij}(\boldsymbol{x}) \Big]$$

Here $\boldsymbol{x} \in \mathbb{R}^3$, $\phi(\boldsymbol{x})$ is a complex-valued field that describes the superconducting matter, $\boldsymbol{A}$ is the Euclidian vector field that comprises the spatial components of the 4-vector $A^\mu = (A^0, \boldsymbol{A})$, and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2, 3)$. μ and q are coupling constants that characterize the vector potential and its coupling to the matter, and r, c and u are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of $S_{\rm GL}$ with respect to all independent fields. (You may want to double check against what you get from the Landau-Lifshitz method we used in class.)

b) Show that this theory is invariant under gauge transformations $\phi(x) \to \phi(\boldsymbol{x}) \, e^{iq\lambda(\boldsymbol{x})}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\nabla}\lambda(\boldsymbol{x})$.

c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 3b), can be made gauge invariant by coupling $\phi(x)$ to the electromagnetic vector potential $A^\mu(x)$.

   hint: Replace the 4-gradient ∂_μ by $D_\mu = \partial_\mu - iqA_\mu$ and add the Maxwell Lagrangian.

   note: If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)