PHYS 622 Classical Electrodynamics W 2019

Problem Assignment # 6 02/13/2019
due 02/20/2019

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18. **Electrostatics in d dimensions (continued from last week)**

Consider Problem #18 as set up in Problem Assignment #5.

b) Calculate and plot the potential φ and the field $\boldsymbol{E}$ for $d = 2$ for the case of a homogeneously charged disk, $\rho(\boldsymbol{x}) = \rho_0\,\Theta(r_0 - |\boldsymbol{x}|)$.

   hint: It is easiest to proceed as in the 3-d case, see Problem 17.

   note: This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0\,\Theta(x_0^2/4 - x^2)$.

   hint: Integrate Poisson's formula directly. (8 points)

19. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \boldsymbol{\nabla}^2)\varphi(\boldsymbol{x}) = 4\pi\rho(\boldsymbol{x})$$

in terms of an integral.
hint: The answer is a generalization of Poisson's formula.

(3 points)

20. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r}\,Q + \frac{1}{r^3}\,\boldsymbol{x}\cdot\boldsymbol{d} + \frac{1}{r^5}\sum_{i,j} x_i x_j Q_{ij} + \ldots$$

   with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from Problem 20 c), and calculate the quadrupole moments Q_{2m} as defined in ch.2 §3.5.

(10 points)