

25. **Field due to distant charges**

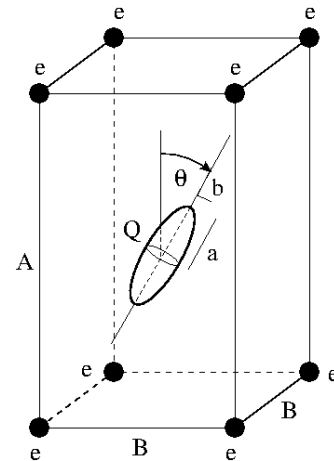
Consider the electric field generated by a charge density  $\rho(\mathbf{y})$  that vanishes inside a sphere with radius  $r_0$ :  $\rho(\mathbf{y}) = 0$  for  $|\mathbf{y}| \leq r_0$ . Show that

- a) If  $\rho$  is invariant under parity operations,  $\rho(-\mathbf{y}) = \rho(\mathbf{y})$ , then the electric field at the origin vanishes.
- b) If  $\rho(\mathbf{y})$  is invariant under rotations about the  $z$ -axis through multiples of an angle  $\alpha$  with  $|\alpha| < \pi$ , then the field-gradient tensor at the origin has the form  $\varphi_{ij}(\mathbf{x} = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$
- c) If  $\rho(\mathbf{y})$  has cubic symmetry, i.e., if  $\rho(\mathbf{y})$  is invariant under rotations through  $\pi/2$  about any of the three axes  $x$ ,  $y$ , and  $z$ , then the field-gradient tensor at the origin vanishes.

(6 points)

26. **Electrostatic interaction**

Consider the following classical model for a nuclear quadrupole moment in a crystal lattice: A rectangular parallelepiped (height  $A$ , length and width  $B$ ) carries a charge  $e$  at each of its eight corners. At the center of the parallelepiped is a homogeneously charged spheroid (charge  $Q$ , semi-axes  $a$  and  $b$ ). The symmetry axis of the spheroid forms an angle  $\theta$  with the  $A$ -axis of the parallelepiped. The center of the spheroid is fixed, but the angle  $\theta$  can vary. Let  $A \gg a$ ,  $B \gg b$ .



- a) Calculate the electrostatic interaction energy  $U$  of this system to quadrupolar order. Show that  $U$  can be expressed in terms of  $e$ , the lattice constants  $A$  and  $B$ , and the quadrupole moment  $Q_{33}$  of the spheroid in the coordinate system of the lattice.
- b) Calculate the quadrupole moment  $Q'_{33}$  of the spheroid in its principal-axes system, and then calculate  $Q_{33}$  by transforming into the lattice system. Express  $U$  as a function of the angle  $\theta$ .  
*hint:* In general, lining up the principal-axes systems would require three Euler angles. However, due to the symmetries of the problem  $Q'_{33}$  and  $Q_{33}$  in the present case are related by only one angle, viz.,  $\theta$ .
- c) Find the equilibrium positions of the spheroid. Make sure to distinguish the cases of prolate and oblate spheroids ( $a > b$  and  $a < b$ , respectively), as well as between the cases  $A > B$  and  $A < B$ .

(15 points)

.../over

### 27. Electric charges in an external field

Consider a static electric charge distribution  $\rho(\mathbf{x})$  subject to a static potential  $\varphi(\mathbf{x})$ . Consider the force  $\mathbf{F}_{\text{el}}$  on the charge distribution and show that  $\mathbf{F}_{\text{el}} = -\nabla U$ , with  $U$  the electrostatic energy calculated in ch.3 §3.6. In particular, convince yourself that the dipole term in the multipole expansion of  $U$  gives the correct potential energy for an electric dipole moment  $\mathbf{d}$  in an electric field  $\mathbf{E}$ .

(3 points)