

33. Wave equations for the electromagnetic fields

Show directly from the Maxwell equations, without introducing potentials, that the fields obey the inhomogeneous wave equations

$$\square \mathbf{E} = -4\pi \left(\nabla \rho + \frac{1}{c^2} \partial_t \mathbf{j} \right) \quad , \quad \square \mathbf{B} = \frac{4\pi}{c} \nabla \times \mathbf{j} .$$

(2 points)

34. Polaritons

As a model for a dielectric, consider a polarization field $\mathbf{P}(\mathbf{x}, t)$ that determines the sources of the electromagnetic fields according to

$$\mathbf{j} = \partial_t \mathbf{P} \quad , \quad \rho = -\nabla \cdot \mathbf{P} \quad .$$

In addition to Maxwell's equations, the dynamics of the system are governed by an equation of motion for \mathbf{P} ,

$$(\partial_t^2 + \omega_0^2) \mathbf{P}(\mathbf{x}, t) = a^2 \mathbf{E}(\mathbf{x}, t) \quad (*) \quad ,$$

where ω_0 is a characteristic frequency and a is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

- Show that Maxwell's equations plus (*) have solutions given by both longitudinal ($\mathbf{k} \parallel \mathbf{E}, \mathbf{P}$) and transverse ($\mathbf{k} \perp \mathbf{E}, \mathbf{P}$) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.
- Show that the transverse waves in the long-wavelength limit are photon-like, viz., $\omega_T(\mathbf{k} \rightarrow 0) = (c/n)|\mathbf{k}|$, and determine the index of refraction n .
- Show that no homogeneous wave propagation is possible in a frequency band $\omega_- < \omega < \omega_+$, and find ω_{\mp} . Derive the Lyddane-Sachs-Teller relation

$$\omega_+^2 / \omega_-^2 = \epsilon(\omega = 0)$$

where $\epsilon(\omega) = 1 + 4\pi a^2 / (\omega_0^2 - \omega^2)$ is the dielectric function of the dielectric.

- Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits $k \rightarrow 0$ and $k \rightarrow \infty$, and plot the result.

(14 points)

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35. Liénard-Wiechert potentials

Consider a point charge e that moves on a given trajectory $\mathbf{X}(t)$ with velocity $\mathbf{v}(t) = \dot{\mathbf{X}}(t)$ which results in charge and current densities

$$\rho(\mathbf{x}, t) = e \delta(\mathbf{x} - \mathbf{X}(t)) \quad , \quad \mathbf{j}(\mathbf{x}, t) = e \mathbf{v}(t) \delta(\mathbf{x} - \mathbf{X}(t))$$

Show that the resulting retarded potentials have the form

$$\varphi(\mathbf{x}, t) = \frac{e}{|\mathbf{x} - \mathbf{X}(t_-)| - \mathbf{v}(t_-) \cdot (\mathbf{x} - \mathbf{X}(t_-))/c}$$
$$\mathbf{A}(\mathbf{x}, t) = \frac{1}{c} \mathbf{v}(t_-) \varphi(\mathbf{x}, t)$$

where t_- is the solution of

$$t_- = t - \frac{1}{c} |\mathbf{x} - \mathbf{X}(t_-)| \quad (*)$$

These are known as Liénard-Wiechert potentials after Alfred-Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

hint: Show that the equation (*) for t_- has one and only one solution.

(6 points)

36. Potential of a uniformly moving charge

Consider a charge e moving uniformly along the x -axis with velocity v : $\mathbf{X}(t) = (vt, 0, 0)$. Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 2 §2.4 by means of a Lorentz transformation.

(6 points)