PHYS 622 Classical Electrodynamics W 2019

Problem Assignment # 1 01/09/2019
due 01/16/2019

---

1. **Dual field tensor**

Show that the dual field tensor $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\lambda\kappa} F_{\lambda\kappa}$ obeys

$$\partial_\mu \tilde{F}^{\mu\nu}(x) = 0$$

(2 points)

2. **Functional derivative**

Let $F[\varphi]$ be a functional of a real-valued function $\varphi(x)$. For simplicity, let $x \in \mathbb{R}$; the generalization to more than one dimension is straightforward. We can (sloppily) define the *functional derivative* of F as

$$\frac{\delta F}{\delta\varphi(x)} := \lim_{\epsilon\to 0} \frac{1}{\epsilon} \Big(F[\varphi(y) + \epsilon\delta(y-x)] - F[\varphi(y)] \Big)$$

a) Calculate $\delta F/\delta\varphi(x)$ for the following functionals:

i) $F = \int dx\, \varphi(x)$

ii) $F = \int dx\, \varphi^2(x)$

iii) $F = \int dx\, (\varphi'(x))^2$ where $\varphi'(x) = d\varphi/dx$
hint: Integrate by parts and assume that the boundary terms vanish.

iv) $F = \int dx\, V(\varphi'(x))$ where V is some given function.

remark: Blindly ignore terms that formally vanish as $\epsilon \to 0$ unless you want to find out why the above definition is problematic. It does work for operational purposes, though.

b) Consider a "Lagrangian" $\mathcal{L}(\varphi(x), \partial_\mu\varphi(x))$ (i.e., a function of φ and its derivatives) and an "action" $S = \int d^4x\, \mathcal{L}$. Show that extremizing S by requiring $\delta S/\delta\varphi(x) \equiv 0$ with the above definition of the functional derivative leads to the Euler-Lagrange equations

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)} = \frac{\partial\mathcal{L}}{\partial\varphi}$$

(3 points)

3. **Massive scalar field**

a) Consider a Lagrangian density

$$\mathcal{L} = \frac{1}{2}\, (\partial_\mu\varphi(x))\, (\partial^\mu\varphi(x)) - \frac{m^2}{2}\, (\varphi(x))^2$$

for a real scalar field $\varphi(x)$. Find the Euler-Lagrange equation for the field φ by requiring $\delta S/\delta\phi(x) = 0$.

b) Generalize this Lagrangian density to a complex field $\phi(x) \in \mathbb{C}$:

$$\mathcal{L} = \frac{1}{2}\, (\partial_\mu\phi(x))\, (\partial^\mu\phi^*(x)) - \frac{m^2}{2}\, |\phi(x)|^2$$

What are the Euler-Lagrange equations now?

c) Consider a local gauge transformation, $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$, with $\Lambda(x)$ a real field that characterizes the transformation. Is the Lagrangian from part b) invariant under such a transformation?

(3 points)

.../over

4. **Ginzburg-Landau theory**

Ginzburg and Landau postulated that superconductivity can be described by an action (which is NOT Lorentz invariant)

$$S_{\mathrm{GL}} = \int d\boldsymbol{x} \Big[r \left|\phi(\boldsymbol{x})\right|^2 + c \left| \left[\nabla - iq\boldsymbol{A}(\boldsymbol{x})\right] \phi(\boldsymbol{x}) \right|^2 + u \left|\phi(\boldsymbol{x})\right|^4 + \frac{1}{16\pi\mu} F_{ij}(\boldsymbol{x}) \, F^{ij}(\boldsymbol{x}) \Big]$$

Here $\boldsymbol{x} \in \mathbb{R}^3$, $\phi(\boldsymbol{x})$ is a complex-valued field that describes the superconducting matter, $\boldsymbol{A}$ is the Euclidian vector field that comprises the spatial components of the 4-vector $A^\mu = (A^0, \boldsymbol{A})$, and $F_{ij} = \partial_i A_j - \partial_j A_i$ $(i, j = 1, 2, 3)$. μ and q are coupling constants that characterize the vector potential and its coupling to the matter, and r, c and u are further parameters of the theory.

a) Find the coupled differential equations (known as Ginzburg-Landau equations) whose solutions extremize this action by considering the functional derivatives of S_{GL} with respect to all independent fields. (You may want to double check against what you get from the Landau-Lifshitz method we used in class.)

b) Show that this theory is invariant under gauge transformations $\phi(x) \to \phi(\boldsymbol{x}) \, e^{iq\lambda(\boldsymbol{x})}$, $\boldsymbol{A}(\boldsymbol{x}) \to \boldsymbol{A}(\boldsymbol{x}) + \boldsymbol{\nabla}\lambda(\boldsymbol{x})$.

c) Show that the Lorentz-invariant Lagrangian density for a massive scalar field, Problem 3b), can be made gauge invariant by coupling $\phi(x)$ to the electromagnetic vector potential $A^\mu(x)$.

   hint: Replace the 4-gradient ∂_μ by $D_\mu = \partial_\mu - iqA_\mu$ and add the Maxwell Lagrangian.

   note: If we had never heard of the electromagnetic potential, insisting on gauge invariance would force us to invent it!

(7 points)

1.) ad 2 §1.1 def. 1 $\rightsquigarrow$ $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$\rightsquigarrow \underline{\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu}} = \partial^\lambda\partial^\mu A^\nu - \partial^\lambda\partial^\nu A^\mu + \partial^\mu\partial^\nu A^\lambda - \partial^\mu\partial^\lambda A^\nu + \partial^\nu\partial^\lambda A^\mu - \partial^\nu\partial^\mu A^\lambda = \underline{0} \quad (*)$$

$\underline{1}$

$$\rightsquigarrow \underline{\partial_\mu \tilde{F}^{\mu\nu}} = \varepsilon^{\mu\nu\kappa\Lambda}\partial_\mu F_{\kappa\Lambda}$$

$$\overset{\text{relabel}}{=} \frac{1}{3}\left[\varepsilon^{\mu\nu\kappa\Lambda}\partial_\mu F_{\kappa\Lambda} + \varepsilon^{\kappa\nu\Lambda\mu}\partial_\kappa F_{\Lambda\mu} + \varepsilon^{\Lambda\nu\mu\kappa}\partial_\Lambda F_{\mu\kappa}\right]$$

$$\overset{\varepsilon\ \text{antisymmetric}}{=} \frac{1}{3}\varepsilon^{\mu\nu\kappa\Lambda}\left[\partial_\mu F_{\kappa\Lambda} + \partial_\kappa F_{\Lambda\mu} + \partial_\Lambda F_{\mu\kappa}\right] \overset{(*)}{=} \underline{\underline{0}} \quad \square$$

$\underline{1}$

2.) a) i) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left(\varphi(y) + \varepsilon \delta(y-x) - \varphi(y) \right) = \int dy\, \delta(y-x) = \underline{\underline{1}}$

ii) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[\left(\varphi(y) + \varepsilon\delta(y-x)\right)^2 - \varphi^2(y) \right]$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[2\varepsilon\varphi(y)\delta(y-x) + O(\varepsilon^2) \right] = \underline{\underline{2\varphi(x)}}$

1

(iii) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dy \left[\left(\varphi'(y) + \varepsilon \frac{d}{dy}\delta(y-x)\right)^2 - (\varphi'(y))^2 \right]$

$= 2\int dy\, \varphi'(y) \frac{d}{dy} \delta(y-x) = \underline{\underline{-2\varphi''(x)}}$

(iv) $\underline{\underline{\frac{\delta F}{\delta\varphi(x)}}} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dx \left[V\left(\varphi'(y) + \varepsilon\frac{d}{dy}\delta(y-x)\right) - V(\varphi'(y)) \right]$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int dx \left[\varepsilon V'(\varphi'(y)) \frac{d}{dy}\delta(y-x) + O(\varepsilon^2) \right]$

$= \underline{\underline{- V''(\varphi'(x))\,\varphi''(x)}}$

1

b) $\underline{\underline{0}} \overset{!}{=} \frac{\delta}{\delta\varphi(x)} \int d^4y\, \mathcal{L}(\varphi(y), \partial_\mu\varphi(y))$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int d^4y \left[\mathcal{L}(\varphi(y) + \varepsilon\delta(y-x), \partial_\mu\varphi(y) + \varepsilon\partial_\mu\delta(y-x)) - \mathcal{L}(\varphi(y), \partial_\mu\varphi(y)) \right.$

$= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int d^4y \left[\varepsilon\delta(y-x) \frac{\partial\mathcal{L}}{\partial\varphi(y)} + \varepsilon(\partial_\mu\delta(y-x)) \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(y))} + O(\varepsilon^2) \right]$

$= \underline{\underline{\frac{\partial\mathcal{L}}{\partial\varphi(x)} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi(x))}}}$ ✓

1

3.) a) $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial}{\partial(\partial_\mu \varphi)} \frac{1}{2} (\partial_\nu \varphi)(\partial_\kappa \varphi) g^{\kappa\nu} = \partial^\mu \varphi$

$\frac{\partial \mathcal{L}}{\partial \varphi} = -m^2 \varphi$

$\rightsquigarrow$ The EL eq. is $(\partial_\mu \partial^\mu + m^2)\,\varphi(x) = 0$

1

b) Treat $\phi(x)$ and $\phi^*(x)$ as independent fields.
Minimizing with respect to ϕ^* yields

$$(\partial_\mu \partial^\mu + m^2)\,\phi(x) = 0$$

and minimizing with respect to ϕ just yields the c.c.:

$$(\partial_\mu \partial^\mu + m^2)\,\phi^*(x) = 0$$

1

c) Under $\phi(x) \to \phi(x)\, e^{i\Lambda(x)}$ we have

$|\phi(x)|^2 \to |\phi(x)|^2$

and

$$\partial_\mu \phi(x) \to (\partial_\mu \phi(x))\, e^{i\Lambda(x)} + i(\partial_\mu \Lambda(x))\,\phi(x)\, e^{i\Lambda(x)}$$

$$\partial^\mu \phi^*(x) \to (\partial^\mu \phi^*(x))\, e^{-i\Lambda(x)} - i(\partial^\mu \Lambda(x))\,\phi^*(x)\, e^{-i\Lambda(x)}$$

$$\to \partial_\mu \phi(x)\, \partial^\mu \phi^*(x) \to \partial_\mu \phi(x)\,\partial^\mu \phi^*(x) - i\,(\partial^\mu \Lambda(x))(\partial_\mu \phi(x))\,\phi^*(x) + i\,(\partial_\mu \Lambda(x))(\partial^\mu \phi^*(x))\,\phi(x) + (\partial_\mu \Lambda(x))(\partial^\mu \Lambda(x))\,|\phi(x)|^2$$

$$\neq \partial_\mu \phi\, \partial^\mu \phi^*$$

$\to$ The Lagrangian is not invariant

1

4.)

$$\boxed{S_{GL} = \int d\vec{x} \left[r|\phi(\vec{x})|^2 + c\left|(\vec{\nabla} - ig\vec{A}(\vec{x}))\phi(\vec{x})\right|^2 + u|\phi(\vec{x})|^4 + \frac{1}{16\pi\mu} F_{ij}(\vec{x})F^{ij}(\vec{x}) \right.}$$

c)

$$0 \stackrel{!}{=} \frac{\delta S_{GL}}{\delta \phi^*(\vec{x})} = r\phi(\vec{x}) + 2u\phi(\vec{x})|\phi(\vec{x})|^2 + c\frac{\delta}{\delta\phi^*(x)}\int d\vec{x}\,(\vec{\nabla} - ig\vec{A})\phi\,(\vec{\nabla} + ig\vec{A})\phi$$

$$= r\phi + 2u\phi|\phi|^2 - c\vec{\nabla}(\vec{\nabla} - ig\vec{A})\phi + ig\vec{A}(\vec{\nabla} - ig\vec{A})\phi$$

1 $$= r\phi + 2u\phi|\phi|^2 - c(\vec{\nabla} - ig\vec{A})^2\phi$$

$$0 \stackrel{!}{=} \frac{\delta S'_{GL}}{\partial \vec{A}(\vec{x})} = c(-ig)\phi(\vec{\nabla} + ig\vec{A})\phi^* + cig\left((\vec{\nabla} - ig\vec{A})\phi\right)\phi^*$$

1 $$+ \frac{1}{16\pi\mu}\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\, F_{ij}(\vec{y})F^{ij}(\vec{y})$$

$$F_{ij}F^{ij} = (\partial_i A_j - \partial_j A_i)(\partial^i A^j - \partial^j A^i) = 2\varepsilon^{kij}\varepsilon_{klm}\partial_i A_j \partial^l A^m$$

$$= 2(\vec{\nabla} \times \vec{A})^2$$

$$\rightarrow \frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\, F_{ij}F^{ij} = -\frac{\delta}{\delta\vec{A}(\vec{x})}\int d\vec{y}\, \vec{A}(\vec{y})\cdot\left(\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{y}))\right)$$

1 $$= -2\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x}))$$

$$= cig\,\phi^*(\vec{\nabla} - ig\vec{A})\phi + \text{c.c.} - \frac{1}{4\pi\mu}\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x}))$$

$$\rightarrow \boxed{\begin{aligned} &-c\left[\vec{\nabla} - ig\vec{A}(\vec{x})\right]^2\phi(\vec{x}) + \left[r + 2u|\phi(\vec{x})|^2\right]\phi(\vec{x}) = 0 \\ &\vec{\nabla}\times(\vec{\nabla}\times\vec{A}(\vec{x})) = 4\pi\mu c i g\,\phi^*(\vec{x})\left[\vec{\nabla} - ig\vec{A}(\vec{x})\right]\phi(\vec{x}) + \text{c.c.} \end{aligned}}$$

1

GL eqs.

b) let $\phi(\vec{x}) \to \phi(\vec{x})\, e^{ig\lambda(\vec{x})}$, $\vec{A}(\vec{x}) \to \vec{A}(\vec{x}) + \vec{\nabla}\lambda(\vec{x})$

$\to \underline{|\phi(\vec{x})|^2} \to \underline{|\phi(\vec{x})|^2}$

and $\underline{F_{ij}(\vec{x})} = \partial_i A_j - \partial_j A_i \to \partial_i (A_j + \cancel{\partial_j \lambda}) - \partial_j (A_i + \cancel{\partial_i \lambda}) = \underline{F_{ij}(\vec{x})}$

1 Finally,

$(\vec{\nabla} - ig\vec{A})\phi \to (\vec{\nabla} - ig\vec{A} - ig\vec{\nabla}\lambda)\,\phi\, e^{ig\lambda} =$

$= (\vec{\nabla}\phi)\, e^{ig\lambda} + \cancel{ig(\vec{\nabla}\lambda)\phi\, e^{ig\lambda}} - ig\vec{A}\phi\, e^{ig\lambda} - \cancel{ig(\vec{\nabla}\lambda)\phi\, e^{ig\lambda}}$

$= e^{ig\lambda} (\vec{\nabla} - ig\vec{A})\phi$

$\to \underline{|(\vec{\nabla} - ig\vec{A})\phi|^2 \to |(\vec{\nabla} - ig\vec{A})\phi|^2}$

1 $\to$ $\underline{S_{GL} \text{ is gauge invariant}}$

c) modify $\mathcal{L}$ from Problem 3b) to read

$$\boxed{\mathcal{L} = (D_\mu \phi(x))(D^\mu \phi(x))^* - m^2 |\phi(x)|^2 - \frac{1}{16\pi} F_{\mu\nu}(x) F^{\mu\nu}(x)} \quad (*)$$

where $D_\mu = \partial_\mu - igA_\mu$

let $\phi(x) \to \phi(x)\, e^{ig\lambda(x)}$, $A^\mu(x) \to A^\mu(x) + \partial^\mu \lambda(x)$

$\to |\phi|^2 \to |\phi|^2$ and $F_{\mu\nu}F^{\mu\nu} \to F_{\mu\nu}F^{\mu\nu}$

$D_\mu \phi \to (\partial_\mu - igA_\mu - ig\partial_\mu \lambda)\,\phi\, e^{ig\lambda}$

$= e^{ig\lambda} (\partial_\mu + \cancel{ig\partial_\mu \lambda} - igA_\mu - \cancel{ig\partial_\mu \lambda})\phi = e^{ig\lambda} D_\mu \phi$

$\to \underline{(D_\mu \phi)(D^\mu \phi)^* \to (D_\mu \phi)(D^\mu \phi^*)}$

1 $\to$ $\underline{(*) \text{ is gauge invariant}}$