Problem Assignment # 3

01/23/2019
due 01/30/2019

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9. **Particle in homogeneous $\boldsymbol{E}$ and $\boldsymbol{B}$ fields**

Consider a point particle (mass m, charge e) in homogeneous fields $\boldsymbol{B} = (0, 0, B)$ and $\boldsymbol{E} = (0, E_y, E_z)$. Treat the motion of the particle nonrelativistically.

a) Show that the motion in z-direction decouples from the motion in the x-y plane, and find $z(t)$.

b) Consider $\xi := x + iy$. Find the equation of motion for ξ, and its most general solution.

hint: Define the *cyclotron frequency* $\omega = eB/mc$, and remember how to solve inhomogeneous ODEs.

c) Show that the time-averaged velocity perpendicular to the plane defined by $\boldsymbol{B}$ and $\boldsymbol{E}$ is given by the *drift velocity*

$$\langle \boldsymbol{v} \rangle = c\, \boldsymbol{E} \times \boldsymbol{B}/\boldsymbol{B}^2$$

Show that $E_y/B \ll 1$ is necessary and sufficient for the non relativistic approximation to be valid.

d) Show that the path projected onto the x-y plane can have three qualitatively different shapes, and plot a representative example for each.

(6 points)

10. **Harmonic oscillator coupled to a magnetic field**

Consider a charged 3-d classical harmonic oscillator (oscillator frequency ω_0, charge e). Put the oscillator in a homogeneous time-independent magnetic field $\boldsymbol{B} = (0, 0, B)$. Show that the motion remains oscillatory, and find the oscillation frequencies in the directions parallel and perpendicular, respectively, to $\boldsymbol{B}$.

(4 points)

11. **Relativistic motion in parallel electric and magnetic fields**

Consider a relativistic charged particle (mass m, charge e) in parallel homogeneous electric and magnetic fields $\boldsymbol{E} = (0, 0, E)$, $\boldsymbol{B} = (0, 0, B)$.

a) Show that the equation of motion for the z-component of the momentum p_z decouples from p_x and p_y, and that the momentum perpendicular to the z-axis is a constant of motion: $p_x^2 + p_y^2 \equiv p_\perp^2 = \text{const.}$

b) Choose the zero of time such that $p_z(t = 0) = 0$, and show that with a suitable chosen origin the z-component of the particle's position can be written

$$z(t) = \frac{1}{eE} \sqrt{T_0^2 + c^2 e^2 E^2 t^2}$$

where T_0 is the kinetic energy (i.e., the energy of the particle without the potential energy due to the fields) at time $t = 0$.

hint: If you have trouble, recall Einstein's law of falling bodies from PHYS 611. You can find my version at http://pages.uoregon.edu/dbelitz/teaching/2013_14/PHYS_611-4/ , Assignment # 5, Problem 21.

.../over

c) Introduce a parameter φ via $d\varphi/dt = ceB/T(t)$, with $T(t)$ the time-dependent kinetic energy. Show that the orbit of the particle can be represented in the parametric form

$$x = \frac{cp_\perp}{eB} \sin\varphi \quad , \quad y = \frac{cp_\perp}{eB} \cos\varphi \quad , \quad z = \frac{T_0}{eE} \cosh(E\varphi/B)$$

and explicitly find the relation between φ and t.

hint: Consider $\pi := p_x + ip_y$ and note that $|\pi| = p_\perp =$ const. by the result of part a).

d) Describe and visualize the orbit, and discuss the motion in the limits of large and small times.

(14 points)

9.) a) Eq. of motion: $m\dot{\vec{v}} = e\vec{E} + \frac{e}{c}\vec{v}\times\vec{B}$

let $\vec{B} = (0,0,B)$ and $\vec{E} = (0,E_y,E_z)$

$$\rightarrow \quad m\ddot{x} = \frac{e}{c}\dot{y}B \quad (1)$$

$$m\ddot{y} = eE_y - \frac{e}{c}\dot{x}B \quad (2)$$

$$m\ddot{z} = eE_z \quad (3)$$

(1) $(3) \rightarrow z(t) = z_0 + v_z^0 t + \frac{eE_z}{2m}t^2$

b) Define $\xi := x + iy$

$(1)+i\cdot(2) \rightarrow m\ddot{\xi} = ieE_y - i\frac{eB}{c}\dot{\xi}$

Define $\omega := \frac{eB}{mc}$ cyclotron frequency

$$\rightarrow \quad \ddot{\xi} + i\omega\dot{\xi} = i\frac{e}{m}E_y \quad (*)$$

Special solution of inhomogeneous eq.: $\dot{\xi} = \frac{eE_y}{m\omega}$

General solution of homogeneous eq.: $\dot{\xi} = a e^{-i\omega t}$ $(a \in \mathbb{C})$

(1) $\rightarrow \dot{\xi}(t) = a e^{-i\omega t} + eE_y/m\omega$ is the most general solution of $(*)$.

c) With $a = b e^{i\alpha}$, $b, \alpha \in \mathbb{R}$

$\rightarrow \dot{\xi} = b e^{-i(\omega-\alpha)t} + eE_y/(m\omega)$

$\rightarrow \alpha$ just shifts the zero of time $\rightarrow \alpha = 0$ w.l.o.g.

$\rightsquigarrow \quad \dot{x} + i\dot{y} = b\cos\omega t - ib\sin\omega t + eE_y/m\omega$

$\rightsquigarrow$

$$\boxed{\begin{aligned} \dot{x} &= b\cos\omega t + eE_y/m\omega \\ \dot{y} &= -b\sin\omega t \end{aligned}} \quad (**)$$

(1)

$\rightsquigarrow \quad \underline{\langle \dot{y} \rangle = 0}, \quad \underline{\langle \dot{x} \rangle} = eE_y/m\omega = \underline{cE_y/B}$ time-averaged velocity

$$= \frac{cE_y B}{B^2} = \underline{c(\vec{E}\times\vec{B})_x / B^2}$$

in general:

$$\underline{\langle \vec{v} \rangle = \frac{c}{B^2}\,\vec{E}\times\vec{B}} \qquad \underline{\text{drift}}\ \underline{\text{velocity}}$$

condition for $v \ll c$: $\underline{E_y/B \ll 1}$ necessary and sufficient condition for non-relativistic approximation

(1)

d) Choose $x(t=0) = 0 = y(t=0)$ w.l.o.g.

$(**) \rightsquigarrow$

$$\boxed{\begin{aligned} x(t) &= \frac{b}{\omega}\sin\omega t + \frac{cE_y}{B}t \\ y(t) &= \frac{b}{\omega}(\cos\omega t - 1) \end{aligned}}$$

$\rightsquigarrow$ The path is a trochoid.

To visualize it, put $\underline{\omega = 1}$ and define $C = cE_y/B$.

$$\rightsquigarrow \quad \boxed{\begin{aligned} x(t) &= b\sin t + Ct \\ y(t) &= b(\cos t - 1) \end{aligned}}$$

This is the projection of the path onto the x-y plane

(1)

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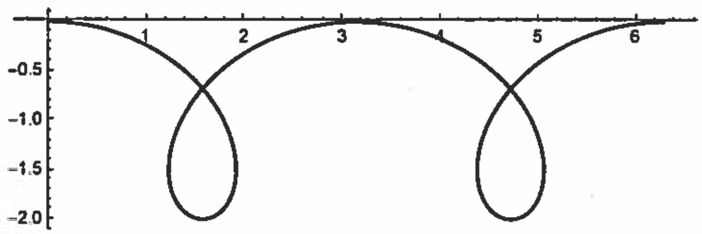
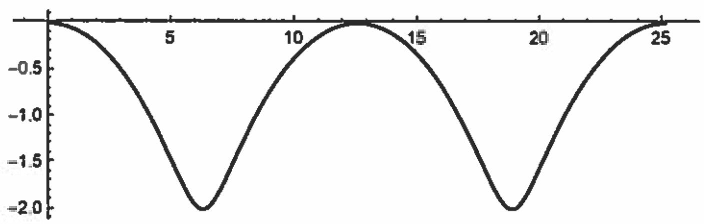
For C < b the trochoid has loops :

In[21]:=
```
b = 1;
c = 0.5;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}]
```

Out[25]=


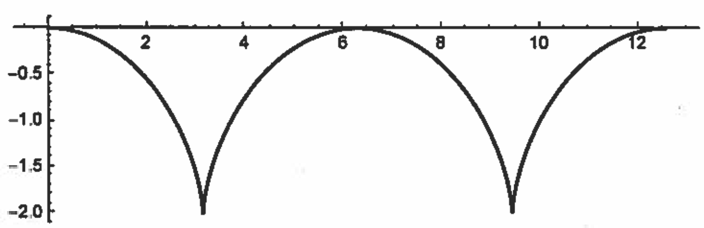
For C > b it does not :

In[86]:=
```
b = 1;
c = 2;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[90]=


And for C = b it degenerates into a cycloid :

In[91]:=
```
b = 1;
c = 1;
x[t_] := b Sin[t] + c t
y[t_] := b (Cos[t] - 1)
ParametricPlot[{x[t], y[t]}, {t, 0, 4 Pi}, AspectRatio → 0.3]
```

Out[95]=


10.) In addition to the restoring force $-m\omega_0^2 x$, the particle is subject to a Lorentz force

$$\frac{e}{c}\vec{v}\times\vec{B} = \frac{e}{c}\left(\dot{y}B, -\dot{x}B, 0\right)$$

→ The eqs of motion are

$$\ddot{x} + \omega_0^2 x = \Omega \dot{y} \qquad (1)$$

$$\ddot{y} + \omega_0^2 y = -\Omega \dot{x} \qquad (2)$$

$$\ddot{z} + \omega_0^2 z = 0 \qquad (3)$$

(1) with $\Omega := eB/mc$ the cyclotron frequency.

(3) → For oscillations in the z-direction, the frequency

(1) $\omega = \omega_0$ is unchanged

Define $\xi := x + iy$ and write

(1)+i·(2) → $\ddot{\xi} + \omega_0^2 \xi = -i\Omega\dot{\xi}$

(1)

ansatz: $\xi(t) = \xi_0 e^{i\omega t}$

→ $-\omega^2 + \omega_0^2 = \omega\Omega$

or $\omega^2 + \Omega\omega - \omega_0^2 = 0$

→ $\omega = \frac{1}{2}\left(-\Omega \pm \sqrt{\Omega^2 + 4\omega_0^2}\right) = \pm\sqrt{\omega_0^2 + \Omega^2/4} - \Omega/2$

→ The motion in the x-y plane is oscillatory, and the eigenfrequencies are

(1) $\omega_\pm = \sqrt{\omega_0^2 + \Omega^2/4} \pm \Omega/2$

11.) a) The Lagrangian is

$$L = L_0 + eEz + \frac{e}{c}\vec{v}\cdot\vec{A}$$

with $L_0 = -mc^2\sqrt{1-v^2/c^2}$

and $\vec{A} = \frac{1}{2}\begin{pmatrix} -By \\ Bx \\ 0 \end{pmatrix} \rightarrow \vec{\nabla}\times\vec{A} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}$ ✓

$\rightarrow \partial L/\partial z = eE$ independent of B

(1) $\rightarrow$ $\boxed{\dot{p}_z = eE}$ decoupled from p_x, p_y

$\rightarrow$ $p_z(t) = eEt$ (with $t=0$ chosen such that $p_z(t=0)=0$),

The forces in x and y-direction are

$$F_x = \frac{e}{c}(\vec{v}\times\vec{B})_x = \frac{eB}{c}v_y \,,\quad F_y = \frac{e}{c}(\vec{v}\times\vec{B})_y = -\frac{eB}{c}v_x$$

(1) $\rightarrow$ $\boxed{\dot{p}_x = \frac{eB}{c}v_y \,,\quad \dot{p}_y = -\frac{eB}{c}v_x}$ (*)

where $\vec{p} = (p_x, p_y, p_z) = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1-v^2/c^2}}$ is the momentum

$$\rightarrow \frac{d}{dt}(p_x^2+p_y^2) = 2(p_x\dot{p}_x + p_y\dot{p}_y) = \frac{2meB}{c}\frac{1}{\sqrt{1-v^2/c^2}}(v_xv_y - v_yv_x) = 0$$

(1) $\rightarrow$ $p_x^2 + p_y^2 =: p_\perp^2 = \text{const}$

b) The kinetic energy is

$$T = \vec{v}\frac{\partial L_0}{\partial \vec{v}} - L_0 = \frac{mc^2}{\sqrt{1-v^2/c^2}} = \sqrt{m^2c^4 + c^2p^2}$$

$$= \sqrt{m^2c^4 + c^2p_\perp^2 + c^2p_z^2} = \sqrt{T_0^2 + c^2e^2E^2t^2}$$

(1) where $T_0 = \sqrt{m^2c^4 + c^2p_\perp^2} = T(t=0)$

But $\vec{v} = \frac{\vec{p}}{\sqrt{m^2+p^2/c^2}} = \frac{c\vec{p}}{\sqrt{m^2c^2+p^2}} = \frac{c^2\vec{p}}{\Gamma}$ (**)

(1) $\rightarrow \dot{z} = \frac{c^2 p_z}{\Gamma} = \frac{c^2 eEt}{\sqrt{\Gamma_0^2+c^2e^2E^2t^2}}$

$\rightarrow \underline{\underline{z(t)}} = z_0 + c^2eE\int_0^t d\tau \frac{\tau}{\sqrt{\Gamma_0^2+c^2e^2E^2\tau^2}} = z_0 + \frac{c^2eE}{c^2e^2E^2}\frac{1}{2}\int_0^{ceEt} \frac{dx}{\sqrt{\Gamma_0^2+x}}$

$= z_0 + \frac{1}{eE}\frac{1}{2}\int_0^{ceEt/\Gamma_0} \frac{dx}{\sqrt{1+x}} = z_0 + \frac{\Gamma_0}{eE}\sqrt{1+c^2e^2E^2t^2/\Gamma_0^2} - \frac{\Gamma_0}{eE}$

(1) $= \underline{\underline{\frac{1}{eE}\sqrt{\Gamma_0^2+c^2e^2E^2t^2}}}$ with $\underline{z_0 = \Gamma_0/eE}$

cf PHYS 611 Problem 21 ✓

c) Define $\tilde{\pi} = p_x + i p_y$

(*) $\rightarrow \dot{\tilde{\pi}} = \frac{eB}{c}(-i)(v_x+iv_y) \overset{(**)}{=} \frac{eB}{c}(-i)\frac{c^2}{\Gamma}\tilde{\pi} = -i\frac{eBc}{\Gamma}\tilde{\pi}$

Define φ by $\frac{eBc}{\Gamma}dt = d\varphi$

(1) $\rightarrow \frac{d\tilde{\pi}}{d\varphi} = -i\tilde{\pi} \quad \rightarrow \underline{\tilde{\pi} = p_\perp e^{-i\varphi}}$

$\rightarrow \tilde{\pi} = \underline{p_\perp e^{-i\varphi}} \overset{(**)}{=} \frac{\Gamma}{c^2}(v_x+iv_y) = \frac{\Gamma}{c^2}\frac{d}{dt}(x+iy) = \frac{\Gamma}{c^2}\frac{eBc}{\Gamma}\frac{d}{d\varphi}(x+iy)$

$= \frac{eB}{c}\frac{d}{d\varphi}(x+iy)$

$\rightarrow \frac{dx}{d\varphi} = \frac{cp_\perp}{eB}\cos\varphi \quad , \quad \frac{dy}{d\varphi} = -\frac{cp_\perp}{eB}\sin\varphi$

(1) $\rightarrow \boxed{x = \frac{cp_\perp}{eB}\sin\varphi \quad , \quad y = \frac{cp_\perp}{eB}\cos\varphi}$ with a suitably chosen origin

that $\frac{d\varphi}{dt} = \frac{eBc}{T} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$

$\rightarrow \quad \varphi = \frac{B}{E}\,\mathrm{arsinh}\left(\frac{ceEt}{T_0}\right)$

check: $\frac{d\varphi}{dt} = \frac{B}{E}\,\frac{ceE}{T_0}\,\frac{1}{\sqrt{1+\left(\frac{ceEt}{T_0}\right)^2}} = \frac{eBc}{\sqrt{T_0^2+c^2e^2E^2t^2}}$ ✓

$\rightarrow \quad \cosh\frac{E\varphi}{B} = \cosh\,\mathrm{arsinh}\left(\frac{ceEt}{T_0}\right) = \sqrt{1+c^2e^2E^2t^2/T_0^2} = T/T_0$

while b) $\rightarrow \quad z = T/eE$

$\rightarrow \quad z = \frac{T_0}{eE}\cosh\frac{E\varphi}{B}$

(1)

Now we have the orbit in a parametric representation:

$$x = \frac{cp_\perp}{eB}\sin\varphi\,, \quad y = \frac{cp_\perp}{eB}\cos\varphi\,, \quad z = \frac{T_0}{eE}\cosh\frac{E\varphi}{B}$$

where φ is related to t via

$$\varphi = \frac{B}{E}\,\mathrm{arsinh}\,\frac{ceEt}{T_0} \quad \text{or} \quad t = \frac{T_0}{ceE}\sinh\frac{E\varphi}{B}$$

(1)

d). The orbit is a helix whose pitch increases exponentially with increasing angle. Choosing the unit of length such that $r_\perp := cp_\perp/eB = 1$ we have

$$x = \sin\varphi\,, \quad y = \cos\varphi\,, \quad z = z_0\cosh(E\varphi/B)$$ with $z_0 = \frac{T_0 B}{cp_\perp E}$

(1)

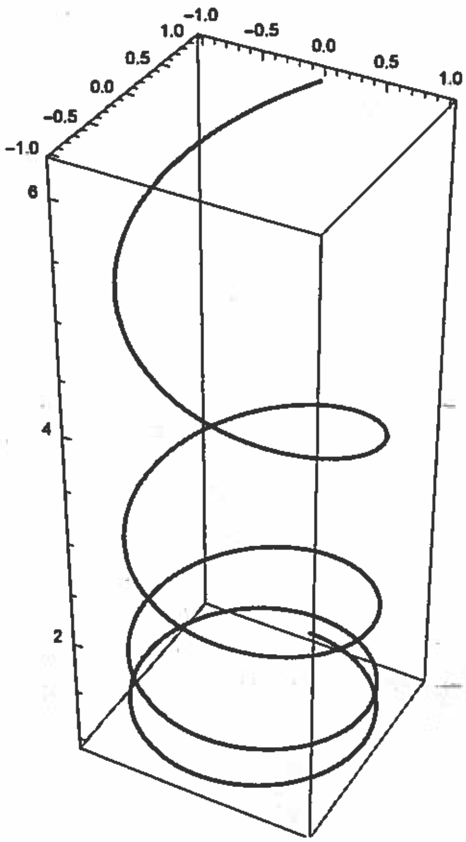
Here is an example for $z_0 = 1$, $E/B = 0.1$:

In[198]:=
```
z0 = 1
EB = 0.1
x[phi_] := Sin[phi]
y[phi_] := Cos[phi]
z[phi_] := z0 Cosh[EB phi]
ParametricPlot3D[{x[phi], y[phi], z[phi]}, {phi, 0, 8 Pi}]
```

Out[198]= 1

Out[199]= 0.1

Out[203]=



(1)

For $t \gg \frac{T_0}{ceE}$ we have $T \to ceEt$

$\to \dot{\varphi} = \frac{ceB}{T} \to \frac{ceB}{ceE}\frac{1}{t} = \frac{B}{E}\frac{1}{t} \to 0$ the angular velocity goes to zero

(1) $z(t) \to ct \to \dot{z} \to c$ the velocity in z-direction approaches c

For $t \ll \frac{T_0}{ceE}$ we have $\dot{\varphi} = eBc/T_0 + O(t^2)$ const. angular velocity

and $z(t) = z_0 + \frac{1}{2}\frac{c^2 e}{T_0}t^2 + O(t^4)$ Galilean result

(1)