Problem Assignment # 5

02/06/2019
due 02/13/2019

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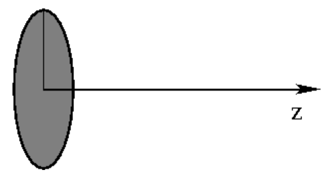
16. **Planar charge distributions**

a) Consider a homogeneously charged infinitesimally thin ring with radius R and total charge Q that is oriented perpendicular to the z-axis. Calculate the electric field on the z-axis.



b) The same for a homogeneously charged disk with charge density σ and radius R. Consider the limits $z \to \infty$, $z \to 0$, and $R \to \infty$, and ascertain that they makes sense.

(4 points)

17. **Spherically symmetric charge distributions**

Consider a spherically symmetric static charge distribution (in spherical coordinates): $\rho(\boldsymbol{x}) = \rho(r)$.

a) Express the electric field in terms of a one-dimensional integral over $\rho(r)$, and the electrostatic potential by a one-dimensional integral over the field.

   hint: Make an *ansatz* for a purely radial field, $\boldsymbol{E}(\boldsymbol{x}) = E(r)\,\hat{e}_r$, and integrate Gauss's law over a spherical volume.

Explicitly calculate and plot the field $\boldsymbol{E}(\boldsymbol{x})$ and the potential $\varphi(\boldsymbol{x})$ for

b) a homogeneously charged sphere

$$\rho(\boldsymbol{x}) = \begin{cases} \rho_0 & \text{if } r \le r_0 \\ 0 & \text{if } r > r_0 \ . \end{cases}$$

c) a homogeneously charged spherical shell

$$\rho(\boldsymbol{x}) = \sigma_0\,\delta(r - r_0) \ .$$

(8 points)

.../over

18. **Electrostatics in d dimensions (to be continued next week)**

Consider the third Maxwell equation in d dimensions:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{x}) = S_d\, \rho(\boldsymbol{x})$$

with the electric field $\boldsymbol{E}$ a d-vector, and S_d the area of the $(d-1)$-sphere: $S_{2n} = 2\pi^n/(n-1)!$ and $S_{2n+1} = 2^{2n+1} n! \pi^n/(2n)!$ for even and odd dimensions, respectively. Define a scalar potential $\varphi(\boldsymbol{x})$ in analogy to the $3-d$ case, such that

$$\boldsymbol{E}(\boldsymbol{x}) = -\boldsymbol{\nabla}\varphi(\boldsymbol{x})$$

and consider Poisson's equation

$$\boldsymbol{\nabla}^2\varphi(\boldsymbol{x}) = -S_d\, \rho(\boldsymbol{x})$$

note: Here we consider a generalization of electrostatics to d-dimensional space, NOT a d-dimensional charge distribution embedded in 3-dimensional space.

a) Show that the Green function $G_d(\boldsymbol{x})$ function for Poisson's equation, i.e., the solution of

$$\boldsymbol{\nabla}^2 G_d(\boldsymbol{x}) = -S_d\, \delta(\boldsymbol{x})$$

is given by

$$G_d(\boldsymbol{x}) = \frac{1}{d-2} \frac{1}{|\boldsymbol{x}|^{d-2}}$$

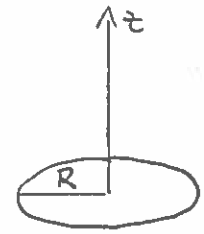
for all $d \neq 2$, and by

$$G_2(\boldsymbol{x}) = \ln(1/|\boldsymbol{x}|)$$

for $d = 2$.

hint: For $d = 1$, differentiate directly, using PHYS 610 Problem 36b). For $d \geq 2$, show that $G_d(\boldsymbol{x})$ is a harmonic function for all $\boldsymbol{x} \neq 0$, then integrate $\boldsymbol{\nabla}^2 G_d$ over a hypersphere around the origin and use Gauss's law.

(4 points)



16.) a) Let the ring be in the z=0 plane:

$$\rho(\vec\xi) = \rho_0\, \delta(\xi_z)\, \delta(r-R)$$

in cylinder coordinates.

Total charge: $\int d\vec\xi\, \rho(\vec\xi) = 2\pi\rho_0 =: Q$

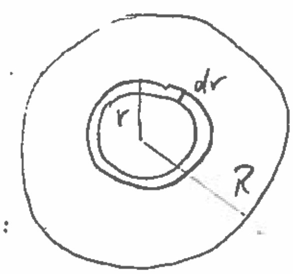
Poisson's formula: $\varphi(\vec x) = \int d\vec\xi\, \frac{\rho(\vec\xi)}{|\vec x-\vec\xi|}$

1 electric field: $\vec E = -\nabla\varphi = -\int d\vec\xi\, \rho(\vec\xi)\, \nabla\frac{1}{|\vec x-\vec\xi|} = \int d\vec\xi\, \rho(\vec\xi)\, \frac{\vec x-\vec\xi}{|\vec x-\vec\xi|^{3/2}}$

symmetry ~> $\vec E(\vec x=(0,0,z)) = E(z)\,\hat z$

1 ~> $E(z) = z\int d\vec\xi\, \frac{\rho(\vec\xi)}{|\vec x-\vec\xi|^{3/2}} = z\int_0^{2\pi} d\varphi\, \frac{\rho_0}{(z^2+R^2)^{3/2}} = \frac{Qz}{(z^2+R^2)^{3/2}}$

b) Charge density: σ



-> charge of ring with radius r, thickness dr:

$$dQ = \sigma\, 2\pi r\, dr = \frac{Q}{\cancel{\pi} R^2} 2\cancel{\pi} r\, dr = \frac{2Q}{R^2} r\, dr \qquad Q = \pi R^2\sigma$$

c) ~> $E(z) = \int_0^R dr\, \frac{2Q}{R^2}\, r\, \frac{z}{(z^2+r^2)^{3/2}} = \frac{2Q}{R^2}\, z\, \frac{1}{2}\int_0^{R^2} \frac{dx}{(z^2+x)^{3/2}}$

1 $$= -\frac{2Q}{R^2}\, \frac{1}{(1+x)^{1/2}}\Big|_0^{R^2/z^2} = \frac{2Q}{R^2}\left(1-\frac{z}{\sqrt{R^2+z^2}}\right) = 2\pi\sigma\left(1-\frac{z}{\sqrt{R^2+z^2}}\right)$$

$E(z\to\infty) = \frac{2Q}{R^2}\left(\cancel{1-1}+\frac{1}{2}\frac{R^2}{z^2}+O(R^4/z^4)\right) = Q/z^2 + O(z^{-4})$ field of point charge

$E(z \to 0) = E(R \to \infty) = 2\pi\sigma$

(1) An infinite sheet with constant charge density produces a field that's independent of z !

17.) a) Consider Gauss's law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

and integrate over a sphere with volume V:

$$\int_V d\vec{x}\,\vec{\nabla}\cdot\vec{E} = \int_{\partial(V)} d\vec{s}\cdot\vec{E} = 4\pi\int_V d\vec{x}\,\rho$$

let $\rho(\vec{x})$ be spherically symmetric, $\rho(\vec{x}) = \rho(r)$, and make an

ansatz: $\boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$ $\quad \hat{e}_r = \vec{x}/|\vec{x}|$

$$\rightsquigarrow \quad 4\pi r^2 E(r) = 4\pi\cdot 4\pi\int_0^r dr'\,r'^2\rho(r')$$

$$\rightsquigarrow \quad \boxed{E(r) = \frac{4\pi}{r^2}\int_0^r dr'\,r'^2\rho(r')}$$

(1)

For the potential, consider $\vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x})$

spherical symmetry $\rightsquigarrow$ $\vec{\nabla}\varphi = \partial_r\varphi\,\hat{r}$

$$\rightsquigarrow \quad E(r) = -\partial_r\varphi(r)$$

$$\rightsquigarrow \quad \varphi(r) = -\int_\infty^r dr'\,E(r') \quad \text{if we choose } \varphi(r=\infty) = 0$$

$$\rightsquigarrow \quad \boxed{\varphi(r) = \int_r^\infty dr'\,E(r')}$$

(1)

b) $E(r) = \frac{4\pi}{r^2}\int_0^r dr'\,r'^2\rho_0\,\Theta(r'<r_0)$

1st case: $r<r_0$ $\quad E(r) = \frac{4\pi\rho_0}{r^2}\int_0^r dx\,x^2 = \frac{4\pi\rho_0}{r^2}\frac{1}{3}r^3 = \frac{4\pi}{3}\rho_0 r$

$$= \frac{4\pi}{3}r_0^3\rho_0\frac{r}{r_0^3} = \frac{Qr}{r_0^3} \quad \text{with } Q = \frac{4\pi}{3}r_0^3\rho_0$$

= total charge

2nd case: $r>r_0$ $\quad E(r) = \frac{4\pi}{r^2}\int_0^{r_0} dr' r'^2 \rho_0 = \frac{4\pi}{r^2}\rho_0 \frac{1}{3} r_0^3 = Q/r^2$

$$\to \boxed{E(r) = \begin{cases} Qr/r_0^3 & \text{for } r \le r_0 \\ Q/r^2 & \text{for } r > r_0 \end{cases}}$$

$$\boxed{\vec{E}(\vec{x}) = E(r)\,\hat{e}_r}$$

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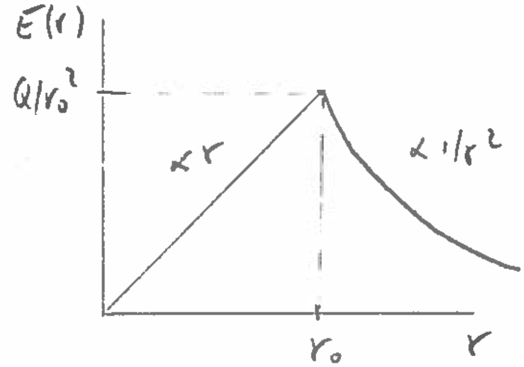
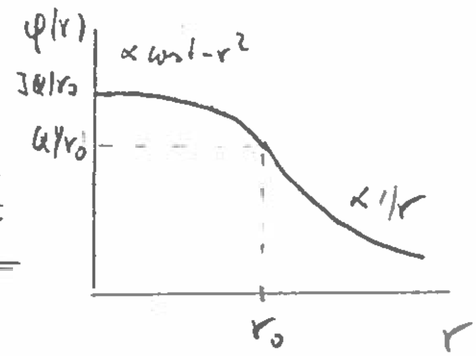


Now the potential:

1st case: $r<r_0$ $\quad \varphi(r) = \int_r^{r_0} dr' \frac{Qr'}{r_0^3} + \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r_0^3}\frac{1}{2}(r_0^2 - r^2) + \frac{Q}{r_0}$

$$= \frac{Q}{2r_0^3}(3r_0^2 - r^2)$$



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2nd case: $r>r_0$ $\quad \varphi(r) = \int_r^{\infty} dr' \frac{Q}{r'^2} = \frac{Q}{r}$

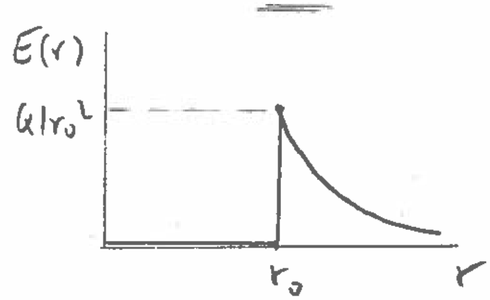
1

$$\boxed{\varphi(r) = \begin{cases} \frac{Q}{2r_0^3}(3r_0^2 - r^2) & \text{for } r<r_0 \\ Q/r & \text{for } r \ge r_0 \end{cases}}$$

c) electric field: $r<r_0$: $\quad E(r) = 0$

$r>r_0$ $\quad E(r) = \frac{4\pi}{r^2}\sigma_0 r_0^2 = \frac{Q}{r^2}$

$$\boxed{E(r) = \begin{cases} 0 & \text{for } r<r_0 \\ Q/r^2 & \text{for } r>r_0 \end{cases}}$$



with $Q = 4\pi r_0^2 \sigma_0$ = total charge

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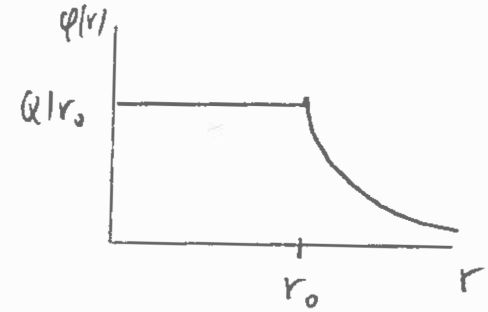
For $r>r_0$, $E(r)$ is the same as for the homogeneous sphere!

potential: $\underline{r < r_0}$ $\quad \varphi(r) = \int_{r_0}^{\infty} dr' \frac{Q}{r'^2} = \underline{\underline{Q/r_0}}$

$\underline{r > r_0}$ $\quad \varphi(r) = \int_{r}^{\infty} dr' \frac{Q}{r'^2} = \underline{\underline{Q/r}}$

(1)

$$\boxed{\varphi(r) = \begin{cases} Q/r_0 & \text{for } r < r_0 \\ Q/r & \text{for } r > r_0 \end{cases}}$$



18. c) Consider $\vec{\nabla}^2 G_d(\vec{x}) = -S_d\,\delta(\vec{x})$

with S_d the surface area of the $(d-1)$-sphere.

proposition: $G_d(\vec{x}) = \frac{1}{d-2}\,\frac{1}{|\vec{x}|^{d-2}}$ for $d \neq 2$

$G_d(\vec{x}) = \log(1/|\vec{x}|)$ for $d = 2$

proof: $d=1$ by direct differentiation:

610 Problem 36 b)

$$\frac{d^2}{dx^2}(-)|x| = -\frac{d}{dx}\operatorname{sgn} x = -2\,\delta(x)$$

$$\rightsquigarrow \frac{d^2}{dx^2} G_{d=1}(x) = \frac{d^2}{dx^2}(-)|x| = -2\,\delta(x) = -S_{d=1}\,\delta(x) \quad \checkmark$$

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$r = |\vec{x}|$

$d=2$: $$\partial_i \partial_j \log|\vec{x}| = \partial_i \frac{x_j}{r^2} = \frac{r^2\delta_{ij} - x_j \frac{2x_i}{r} r}{r^4} = \frac{r^2\delta_{ij} - 2x_i x_j}{r^4}$$

$$\rightsquigarrow \vec{\nabla}^2 \log|\vec{x}| = \partial_i \partial^i \log|\vec{x}| = (2-2)\frac{1}{r^2} = 0 \quad \forall\, r \neq 0$$

1

$\rightsquigarrow$ $\log|\vec{x}|$ is a harmonic fct. $\forall\, \vec{x} \neq 0$

Now integrate over a circle C_0 radius r_0:

$$\int_{C_0} d^2x\, \vec{\nabla}^2 \log|\vec{x}| = \int_{C_0} d^2x\, \vec{\nabla}\cdot(\vec{\nabla}\log r) \overset{\text{Gauss}}{=} \int_{(C_0)} d\vec{\sigma}\cdot\vec{\nabla}\log r$$

$$= \int_0^{2\pi} d\varphi\, r_0 \frac{\vec{x}}{r}\cdot\frac{\vec{x}}{r^2}\Big|_{r_0} = 2\pi$$

$$\rightsquigarrow \vec{\nabla}^2 \log|\vec{x}| = 2\pi\,\delta(\vec{x}) = S_{d=2}\,\delta(\vec{x})$$

1

$$\rightsquigarrow G_{d=2}(\vec{x}) = -\log|\vec{x}| = \log(1/|\vec{x}|)$$

$d>2$: $$\partial_i\partial_j \frac{1}{|\vec x|^{d-2}} = -\frac{d-2}{\cancel{2}}\,\partial_i \frac{\;}{|\vec x|^{d}} = -(d-2)\left(\frac{\delta_{ij}}{|\vec x|^{d/2}} - \frac{d}{2}\frac{\cancel{2}x_i x_j}{|\vec x|^{(d+2)/2}}\right)$$

$$= -(d-2)\,\frac{r^2\delta_{ij} - d\,x_i x_j}{r^{(d+2)/2}}$$

$$\Rightarrow \vec\nabla^2 \frac{1}{|\vec x|^{d-2}} = -(d-2)(d-d)\,\frac{1}{r^{(d-1)/2}} = 0 \quad \forall\, r \neq 0$$

Integrate over a hypersphere S_0 with radius r_0:

$$\underline{\int_{S_0} d^dx\; \vec\nabla^2 \frac{1}{|\vec x|^{d-2}}} = \int_{(S_0)} d\vec\sigma\cdot\vec\nabla \frac{1}{|\vec x|^{d-2}} = S_d\, r_0^{d-1}(-)(d-2)\,\frac{\vec x}{r}\cdot\frac{\vec x}{r^{d}}$$

$$= -(d-2)\,S_d$$

$$\vec\nabla^2 \frac{1}{|\vec x|^{d-2}} = -(d-2)\,S_d\,\delta(\vec x)$$

(1) $$G_{d>2}(\vec x) = \frac{1}{d-2}\,\frac{1}{|\vec x|^{d-2}}$$