PHYS 622	Classical Electrodynamics	W 2019
	Problem Assignment # 6	02/13/2019 due 02/20/2019

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18. **Electrostatics in d dimensions (continued from last week)**

Consider Problem #18 as set up in Problem Assignment #5.

b) Calculate and plot the potential φ and the field $\boldsymbol{E}$ for $d = 2$ for the case of a homogeneously charged disk, $\rho(\boldsymbol{x}) = \rho_0\, \Theta(r_0 - |\boldsymbol{x}|)$.

hint: It is easiest to proceed as in the 3-d case, see Problem 17.

note: This problem plays an important role in the theory of the Kosterlitz-Thouless transition, for which part of the 2016 Nobel prize in Physics was awarded.

c) The same for $d = 1$ for the case of a uniformly charged rod, $\rho(x) = \rho_0\, \Theta(x_0^2/4 - x^2)$.

hint: Integrate Poisson's formula directly. (8 points)

19. **Helmholtz equation**

Find the most general Fourier transformable solution of the Helmholtz equation

$$(\kappa^2 - \boldsymbol{\nabla}^2)\varphi(\boldsymbol{x}) = 4\pi\rho(\boldsymbol{x})$$

in terms of an integral.
hint: The answer is a generalization of Poisson's formula.

(3 points)

20. **Quadrupole moments**

a) Consider a localized charge density as in ch.2 §3.1 and carry the expansion of the potential to $O(1/r^3)$. Show that the potential to that order is given by

$$\varphi(\boldsymbol{x}) = \frac{1}{r}\, Q + \frac{1}{r^3}\, \boldsymbol{x} \cdot \boldsymbol{d} + \frac{1}{r^5} \sum_{i,j} x_i x_j Q_{ij} + \ldots$$

with Q the total charge and $\boldsymbol{d}$ the dipole moment, and determine the quadrupole tensor Q_{ij}.

b) Show that the quadrupole tensor is independent of the choice of the origin provided the total charge and the dipole moment vanish.

c) Consider a homogeneously charged ellipsoid $(x/a)^2 + (y/b)^2 + (z/c)^2 \leq 1$ and calculate the quadrupole tensor Q_{ij} with respect to the ellipsoid's center. Check to make sure that the result for Q_{ij} is traceless.

d) Let the charge density be invariant under rotations about the z-axis through multiples of an angle α, with $|\alpha| < \pi$. Show that in this case the quadrupole tensor has the form $\begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$. Make sure your result from part c) conforms with this for the special case $a = b$.

e) Consider the homogeneously charged ellipsoid from Problem 20 c), and calculate the quadrupole moments Q_{2m} as defined in ch.2 §3.5.

(10 points)

18. c) Consider $\vec{\nabla}^2 G_d(\vec{x}) = -S_d\, \delta(\vec{x})$

with S_d the surface area of the $(d-1)$-sphere.

proposition: $G_d(\vec{x}) = \frac{1}{d-2} \frac{1}{|\vec{x}|^{d-2}}$ for $d \neq 2$

$G_d(\vec{x}) = \log(1/|\vec{x}|)$ for $d = 2$

proof: $d=1$ by direct differentiation: (cf. Problem 36 b)

$$\frac{d^2}{dx^2}(-)|x| = -\frac{d}{dx}\,\mathrm{sgn}\,x = -2\delta(x)$$

$$\rightsquigarrow \frac{d^2}{dx^2} G_{d=1}(x) = \frac{d^2}{dx^2}(-)|x| = -2\delta(x) = -S_{d=1}\,\delta(x) \quad ✓$$

1

$r = |\vec{x}|$

$d=2$: $$\partial_i \partial_j \log|\vec{x}| = \partial_i \frac{x_j}{r^2} = \frac{r^2 \delta_{ij} - x_j \frac{2x_i}{r} r}{r^4} = \frac{r^2\delta_{ij} - 2x_i x_j}{r^4}$$

$$\rightsquigarrow \vec{\nabla}^2 \log|\vec{x}| = \partial_i \partial^i \log|\vec{x}| = (2-2)\frac{1}{r^2} = 0 \quad \forall\, r \neq 0$$

1

$\rightsquigarrow \log|\vec{x}|$ is a harmonic fct. $\forall\, \vec{x} \neq 0$

Now integrate over a circle C_0 radius r_0:

$$\int_{C_0} d^2x\, \vec{\nabla}^2 \log|\vec{x}| = \int_{C_0} d^2x\, \vec{\nabla}\cdot(\vec{\nabla}\log r) \overset{\text{Gauss}}{=} \int_{(C_0)} d\vec{\sigma}\cdot\vec{\nabla}\log r$$

$$= \int_0^{2\pi} d\varphi\, r_0 \frac{\vec{x}}{r}\cdot\frac{\vec{x}}{r^2}\Big|_{r_0} = 2\pi$$

$$\rightsquigarrow \vec{\nabla}^2 \log|\vec{x}| = 2\pi\,\delta(\vec{x}) = S_{d=2}\,\delta(\vec{x})$$

1

$$\rightsquigarrow G_{d=2}(\vec{x}) = -\log|\vec{x}| = \log(1/|\vec{x}|)$$

$d>2$: $\partial_i \partial_j \frac{1}{|\vec{x}|^{d-2}} = -\frac{d-2}{2} \partial_i \frac{2x_j}{|\vec{x}|^d} = -(d-2)\left(\frac{\delta_{ij}}{|\vec{x}|^{d/2}} - \frac{d}{2}\frac{2x_i x_j}{|\vec{x}|^{(d+2)/2}}\right)$

$$= -(d-2)\,\frac{r^2\delta_{ij} - d x_i x_j}{r^{(d+2)/2}}$$

$$\rightsquigarrow \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}} = -(d-2)(d-d)\frac{1}{r^{(d-1)/2}} = 0 \quad \forall\, r \neq 0$$

Integrate over a hypersphere S_0 with radius r_0:

$$\underline{\int_{S_0} d^d x\, \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}}} = \int_{(S_0)} d\vec{\sigma}\cdot\vec{\nabla}\frac{1}{|\vec{x}|^{d-2}} = S_d' r_0^{d-1}(-)(d-2)\frac{\vec{x}}{r}\cdot\frac{\vec{x}}{r^{d}}$$

$$= \underline{-(d-2)\, S_d'}$$

$$\rightsquigarrow \vec{\nabla}^2 \frac{1}{|\vec{x}|^{d-2}} = -(d-2)\, S_d\, \delta(\vec{x})$$

1 $\rightsquigarrow \underline{\underline{G_{d>2}(\vec{x})}} = \underline{\underline{\frac{1}{d-2}\frac{1}{|\vec{x}|^{d-2}}}}$

b) It is easiest to start with the field. Gauss's law in 2-d reads

$$\vec{\nabla}\cdot\vec{E}(\vec{x}) = 2\pi\,\rho(\vec{x})$$

and proceeding as in Problem 17 we have

$$2\pi r\, E(r) = 2\pi\cdot 2\pi \int_0^r dr'\, r'\, \rho(r')$$

for a charge distribution $\rho(\vec{x}) = \rho(r)$ and $\vec{E}(\vec{x}) = E(r)\,\hat{e}_r$.

1 $$\rightsquigarrow \boxed{E(r) = \frac{2\pi}{r}\int_0^r dr'\, r'\, \rho(r')}$$

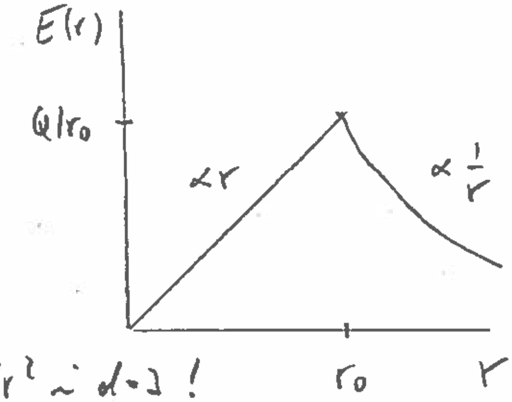
Homogeneously charged disk: $\rho(r) = \rho_0\, \Theta(r_0 - r)$

1st case: $r<r_0$ $\quad E(r) = \frac{2\pi}{r}\int_0^r dr'\,r'\rho_0 = \frac{2\pi\rho_0}{r}\frac{1}{2}r^2 = \pi\rho_0 r$

$= \frac{Q}{r_0^2}r$ $\quad$ with $Q = \pi r_0^2\rho_0$ total charge

2nd case: $r>r_0$ $\quad E(r) = \frac{2\pi}{r}\rho_0\frac{1}{2}r_0^2 = \frac{Q}{r}$

$$\to \boxed{\begin{array}{l} \vec{E}(\vec{x}) = E(r)\,\hat{e}_r \\ E(r) = \begin{cases} Qr/r_0^2 & \text{for } r<r_0 \\ Q/r & \text{for } r>r_0 \end{cases} \end{array}}$$



(1)

Field falls off only as $1/r$, as opposed to $1/r^2$ in $d=3$!

Now the potential: $\quad \vec{E}(\vec{x}) = -\vec{\nabla}\varphi(\vec{x}) = -\partial_r\varphi(r)\,\hat{e}_r$

$\to E(r) = -\partial_r\varphi(r)$

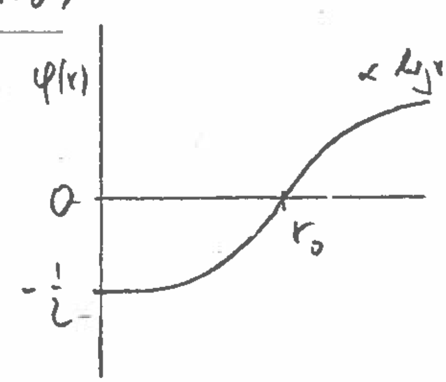
$\to \varphi(r) = -\int_{\text{const}}^r dr'\,E(r')$

$\to \boxed{\varphi(r) = -\int_{r_0}^r dr'\,E(r')}$ $\quad$ with the choice $\varphi(r=r_0)=0$

1st case: $r<r_0$ $\quad \varphi(r) = -\int_r^{r_0} dr'\,\frac{Qr'}{r_0^2} = -\frac{Q}{2r_0^2}(r_0^2-r^2) = \frac{+Q}{2}\left(\frac{r^2}{r_0^2}-1\right)$

2nd case: $r>r_0$ $\quad \varphi(r) = \int_{r_0}^r dr'\,\frac{Q}{r'} = Q\log(r/r_0)$

$$\boxed{\varphi(r) = Q \times \begin{cases} \frac{1}{2}(r^2/r_0^2-1) & \text{for } r<r_0 \\ \log(r/r_0) & \text{for } r>r_0 \end{cases}}$$



(1)

+ (1) for sketches

c) In 1-d it is easiest to integrate Poisson's formula directly:

$$\varphi(x) = \int dy\, G_{d=1}(x-y)\, \rho(y) = -\int dy\, |x-y|\, \rho_0\, \Theta(x_0^2/4 - y^2)$$

$$= -\rho_0 \int_{-x_0/2}^{x_0/2} dy\, |x-y| = \varphi(-x)$$

1

Let $x \geq 0$.

1st case: $x < x_0/2$ $\quad \varphi(x) = -\rho_0 \int_{-x_0/2}^{x} dy\,(x-y) + \rho_0 \int_{x}^{x_0/2} dy\,(x-y)$

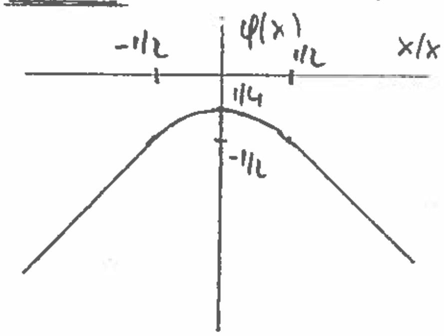
$$= -\rho_0\left[x(\cancel{x}+\tfrac{x_0}{2}) - \tfrac{1}{2}(\cancel{x^2} - \tfrac{1}{4}x_0^2)\right] + \rho_0\left[x(\tfrac{x_0}{2} - \cancel{x}) - \tfrac{1}{2}(\tfrac{x_0^2}{4} - \cancel{x^2})\right]$$

$$= -\rho_0\left(x^2 + \tfrac{1}{4}x_0^2\right)$$

2nd case: $x > x_0/2$ $\quad \varphi(x) = -\rho_0 \int_{-x_0/2}^{x_0/2} dy\,(x-y) = -\rho_0\, x x_0 = -\rho_0 x_0 x$

$= -Qx$ with $Q = \rho_0 x_0$ total charge

$$\boxed{\varphi(x) = -Q x_0 \times \begin{cases} \dfrac{x^2}{x_0^2} + \dfrac{1}{4} & \text{for } |x| < x_0/2 \\ |x|/x_0 & \text{for } |x| > x_0/2 \end{cases}}$$

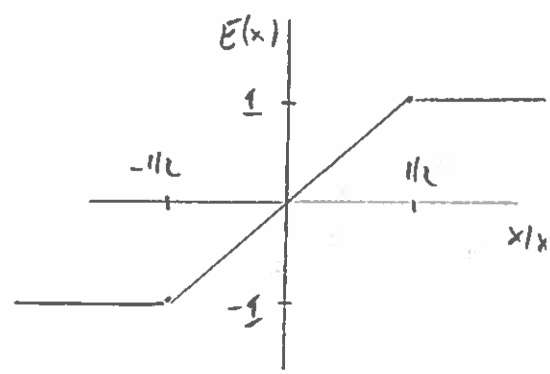


1

Now the field:

$E(x) = -\partial_x \varphi(x) =$

$$\boxed{E(x) = Q \times \begin{cases} 2x/x_0 & \text{for } |x| < x_0/2 \\ \operatorname{sgn} x & \text{for } |x| > x_0/2 \end{cases}}$$



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1

Field does not fall off for $|x| \to \infty$!

[illegible]

19.) Helmholtz eq: $(\vec{k}^2 - \vec{\nabla}^2)\varphi(\vec{x}) = 4\pi\rho(\vec{x})$

Fourier transf. as in ch 3 § 2

$$(\vec{k}^2 + \vec{\ell}^2)\hat{\varphi}(\vec{\ell}) = 4\pi\hat{\rho}(\vec{\ell})$$

(1)

$$\rightarrow \hat{\varphi}(\vec{\ell}) = \frac{4\pi}{k^2 + \vec{\ell}^2}\hat{\rho}(\vec{\ell})$$

$$\rightarrow \varphi(\vec{x}) = \int \frac{d\vec{\ell}}{(2\pi)^3} e^{i\vec{\ell}\vec{x}} \frac{4\pi}{k^2 + \vec{\ell}^2}\hat{\rho}(\vec{\ell})$$

$$= \int d\vec{y}\, v_{sc}(\vec{x} - \vec{y})\,\rho(\vec{y})$$ by the convolution theorem, PHYS 610 ch 2 §7.1

(1)

where $v_{sc}(\vec{x})$ is the Fourier back transf. of the screened Coulomb potential

$$\hat{v}_{sc}(\vec{\ell}) = \frac{4\pi}{k^2 + \vec{\ell}^2}$$

610 Problem 27 b) $\rightarrow$ $v_{sc}(\vec{x}) = \frac{1}{r} e^{-kr}$ with $r = |\vec{x}|$

$$\rightarrow \boxed{\varphi(\vec{x}) = \int d\vec{y}\, \frac{e^{-k|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}\rho(\vec{y})}$$

(1)

For $k=0$ we recover Poisson's formula

20.) a) $\frac{1}{|\vec{x}-\vec{y}|} = \frac{1}{r}\left(1 - 2\frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{y^2}{r^2}\right)^{-1/2} = \frac{1}{r}\left(1 + \frac{\vec{x}\cdot\vec{y}}{r^2} - \frac{1}{2}\frac{y^2}{r^2} + \frac{3}{2}\frac{(\vec{x}\cdot\vec{y})^2}{r^4} + \dots\right)$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{3}{2} x_i x_j y_i y_j \frac{1}{r^4} - \frac{1}{2} y^2 \delta_{ij} x_i x_j \frac{1}{r^4} + \dots\right]$$

$$= \frac{1}{r}\left[1 + \frac{\vec{x}\cdot\vec{y}}{r^2} + \frac{1}{2} x_i x_j (3 y_i y_j - \delta_{ij} y^2)\frac{1}{r^4} + \dots\right]$$

$$\rightsquigarrow \underline{\underline{\varphi(\vec{x})}} = \int d\vec{y}\, \frac{\rho(\vec{y})}{|\vec{x}-\vec{y}|} = \frac{1}{r}\int d\vec{y}\, \rho(\vec{y}) + \frac{1}{r^3}\vec{x}\cdot\int d\vec{y}\, \vec{y}\rho(\vec{y})$$

$$+ \frac{1}{2}\frac{1}{r^5} x_i x_j \int d\vec{y}\, (3 y_i y_j - \delta_{ij} y^2)\rho(\vec{y}) + \dots$$

$$= \underline{\underline{\frac{1}{r} Q + \frac{1}{r^3}\vec{x}\cdot\vec{d} + \frac{1}{r^5}\sum_{ij} x_i x_j Q_{ij} + O(1/r^4)}}$$

where $Q = \int d\vec{y}\, \rho(\vec{y})$ monopole moment

$\vec{d} = \int d\vec{y}\, \vec{y}\rho(\vec{y})$ dipole moment

$\underline{\underline{Q_{ij} = \frac{1}{2}\int d\vec{y}\, (3 y_i y_j - \delta_{ij} y^2)\rho(\vec{y})}}$ quadrupole moment

1

b) $\rho'(\vec{y}) = \rho(\vec{y} - \vec{a})$

$$\rightsquigarrow \underline{\underline{Q'_{ij}}} = \frac{1}{2}\int d\vec{y}\, (3 y_i y_j - \delta_{ij} y^2)\rho'(\vec{y})$$

$$= \frac{1}{2}\int d\vec{y}\left[3(y_i + a_i)(y_j + a_j) - \delta_{ij}(\vec{y}+\vec{a})^2\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{1}{2}\int d\vec{y}\left[3 a_i y_j + 3 a_j y_i + 3 a_i a_j - \delta_{ij}(2\vec{a}\cdot\vec{y} + \vec{a}^2)\right]\rho(\vec{y})$$

$$= Q_{ij} + \frac{3}{2} a_i d_j + \frac{3}{2} a_j d_i + \frac{3}{2} a_i a_j Q - \delta_{ij}\vec{a}\cdot\vec{d} - \delta_{ij}\frac{1}{2}\vec{a}^2 Q$$

$$= Q_{ij} \quad \text{if } \underline{\vec{d} = Q = 0}$$

1

c) ellipsoid: $x^2/a^2+y^2/b^2+z^2/c^2 \le 1$

$$\rightarrow \quad Q_{ij} = \frac{1}{2}\int d\vec{x}\,(3x_ix_j - \delta_{ij}\vec{x}^2)\,\Theta(x^2/a^2+y^2/b^2+z^2/c^2 \le 1)\,\rho$$

where $\vec{x} = (x,y,z)$ and ρ = charge density

(1) symmetry $\rightarrow$ $\underline{Q_{ij} = 0 \text{ unless } i=j}$

$$\underline{\underline{Q_{11}}} = \frac{\rho}{2}\int dx\,dy\,dz\,(2x^2-y^2-z^2)\,\Theta(x^2/a^2+y^2/b^2+z^2/c^2 \le 1)$$

$$= \frac{1}{2}\rho\, abc \int dx\,dy\,dz\,(2a^2x^2-b^2y^2-c^2z^2)\,\Theta(x^2+y^2+z^2 \le 1)$$

$$= \frac{1}{2}\rho\, abc\Big[2a^2\int_0^1 dr\,r^2\int_{-1}^{1} d\xi\int_0^{2\pi} d\varphi\, r^2\sin^2\vartheta\cos^2\varphi$$

$$-b^2\int_0^1 dr\,r^2\int_{-1}^{1} d\xi\int_0^{2\pi} d\varphi\, r^2\sin^2\vartheta\sin^2\varphi$$

$$-c^2\int_0^1 dr\,r^2\int_{-1}^{1} d\xi\int_0^{2\pi} d\varphi\, r^2\cos^2\vartheta\Big]$$

$$= \frac{1}{2}\rho\, abc\Big[2a^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - b^2\pi\left(2-\frac{2}{3}\right)\frac{1}{5} - c^2 2\pi\frac{2}{3}\frac{1}{5}\Big]$$

$$= \frac{1}{2}\rho\, abc\,\frac{\pi}{5}\Big[\frac{8}{3}a^2 - \frac{4}{3}b^2 - \frac{4}{3}c^2\Big]$$

$$= \frac{4\pi}{3}abc\,\rho\,\frac{1}{10}\big[2a^2-b^2-c^2\big]$$

(1) $$= \underline{\underline{Q\,\frac{1}{10}(2a^2-b^2-c^2)}} \qquad \text{with } \underline{Q = \frac{4\pi}{3}\rho\, abc} = \text{total charge}$$

$\underline{\underline{Q_{22}}} = \underline{\underline{Q\,\frac{1}{10}(2b^2-a^2-c^2)}}$ by symmetry

$\underline{\underline{Q_{33}}} = \underline{\underline{Q\,\frac{1}{10}(2c^2-b^2-a^2)}}$ by symmetry

(1) check: $\underline{Q_{11}+Q_{22}+Q_{33} = 0}$ ✓

d) As a real symmetric tensor, Q_{ij} can always be diagonalized

→ The most general form of Q_{ij} in its principal axes system is

$$Q_{ij} = \begin{pmatrix} q_+ + q_- & 0 & 0 \\ 0 & q_+ - q_- & 0 \\ 0 & 0 & -2q_+ \end{pmatrix}$$

where

$$\underline{\underline{q_-}} = \frac{1}{2}\left(Q_{11} - Q_{22}\right) = \frac{1}{2}\int d\vec{x}\, \rho(\vec{x}) \left[2x^2 - y^2 - z^2 - 2y^2 + x^2 + z^2\right]$$

$$= \frac{3}{2}\int d\vec{x}\, \rho(\vec{x}) \left(x^2 - y^2\right)$$

(1)

cylindrical coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$; $x^2 - y^2 = r^2(\cos^2\varphi - \sin^2\varphi) = r^2\cos 2\varphi$

$$\rightarrow q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r \int dz\, \rho(r,\varphi,z)\, r^2 \cos 2\varphi$$

Now let $\rho(r,\varphi,z) = \rho(r,\varphi+\alpha,z)$

$$\rightarrow q_- = \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi+\alpha,z) \cos 2\varphi$$

$$= \frac{3}{2}\int_\alpha^{2\pi+\alpha} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z) \cos 2(\varphi-\alpha)$$

$$= \frac{3}{2}\int_0^{2\pi} d\varphi \int_0^\infty dr\, r^3 \int dz\, \rho(r,\varphi,z) \left(\cos 2\varphi \cos 2\alpha + \sin 2\varphi \sin 2\alpha\right)$$

$\boxed{r^2 \sin 2\varphi \sim r^2 \sin\varphi\cos\varphi = xy \rightarrow \text{the second term is} \propto Q_{12} = 0}$

$$= \cos 2\alpha \cdot q_- \quad \rightarrow \quad \underline{\underline{q_- = 0}} \text{ since } \alpha \neq \pi$$

(1)

part c) with $a = b$ → $Q_{11} = Q_{22} = \frac{Q}{10}(a^2 - c^2)$ ✓

e)

$$Q_{20} = \cancel{\sqrt{\tfrac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\tfrac{5}{4\pi}}}\, \tfrac{1}{2}(3\eta^2-1)$$

$$= \tfrac{1}{2}\int d\vec{x}\, \rho(\vec{x})\,(3z^2-r^2) = D_{33}$$

1

$$Q_{2,\pm 1} \propto \int d\Omega\, \underbrace{\underbrace{\rho(r,\Omega)}_{\text{even}}\, \underbrace{P_2^{\pm 1}(\eta)}_{\text{odd}}}_{\text{fct. of } \eta} = 0 \quad \text{by symmetry}$$

1

$$Q_{22} = \cancel{\sqrt{\tfrac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\tfrac{5}{4\pi}}}\, \frac{1}{\sqrt{4!}}\, e^{2i\varphi}\, 3(1-\eta^2)$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (1-\eta^2)(\cos 2\varphi + i \sin 2\varphi)$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 \sin^2\vartheta\,(\cos^2\varphi - \sin^2\varphi) + \frac{3i}{\sqrt{24}} \int d\vec{x}\, \underbrace{\underbrace{\rho(\vec{x})\, r^2 \sin^2\vartheta}_{\text{even}}\, \underbrace{\sin 2\varphi}_{\text{odd}}}_{\text{fct. of } \varphi \to 0}$$

$$= \frac{3}{\sqrt{24}} \int d\vec{x}\, \rho(\vec{x})\, r^2 (\sin^2\vartheta \cos^2\varphi - \sin^2\vartheta \sin^2\varphi)$$

$$\left.\begin{aligned} x &= r \sin\vartheta \cos\varphi \\ y &= r \sin\vartheta \sin\varphi \\ z &= r\cos\vartheta \end{aligned}\right\} \rightsquigarrow x^2 - y^2 = r^2 \sin^2\vartheta\,(\cos^2\varphi - \sin^2\varphi)$$

$$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x})\,(x^2-y^2)$$

$$= \frac{3}{2\sqrt{6}} \int d\vec{x}\, \rho(\vec{x}) \left[(2x^2-y^2-z^2) - (2y^2-x^2-z^2)\right] \frac{1}{3}$$

$$= \frac{1}{\sqrt{6}} (D_{11} - D_{22})$$

1

$$Q_{2,-2} = \cancel{\sqrt{\tfrac{4\pi}{5}}} \int_0^\infty dr\, r^4 \int d\Omega\, \rho(r,\Omega) \cancel{\sqrt{\tfrac{5}{4\pi}}}\, \sqrt{4!}\, e^{-2i\varphi}\, \frac{1}{8}(1-\eta^2) = Q_{22}$$