Problem Assignment # 7

02/20/2019
due 02/27/2019

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25. **Field due to distant charges**

Consider the electric field generated by a charge density $\rho(\boldsymbol{y})$ that vanishes inside a sphere with radius r_0: $\rho(\boldsymbol{y}) = 0$ for $|\boldsymbol{y}| \leq r_0$. Show that

a) If ρ is invariant under parity operations, $\rho(-\boldsymbol{y}) = \rho(\boldsymbol{y})$, then the electric field at the origin vanishes.

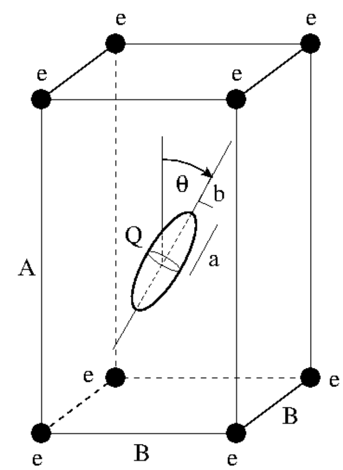
b) If $\rho(\boldsymbol{y})$ is invariant under rotations about the z-axis through multiples of an angle α with $|\alpha| < \pi$, then the field-gradient tensor at the origin has the form $\varphi_{ij}(\boldsymbol{x} = 0) = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix}$

c) If $\rho(\boldsymbol{y})$ has cubic symmetry, i.e., if $\rho(\boldsymbol{y})$ is invariant under rotations through $\pi/2$ about any of the three axes x, y, and z, then the field-gradient tensor at the origin vanishes.

(6 points)

26. **Electrostatic interaction**

Consider the following classical model for a nuclear quadrupole moment in a crystal lattice: A rectangular parallelepiped (height A, length and width B) carries a charge e at each of its eight corners. At the center of the parallelepiped is a homogeneously charged spheroid (charge Q, semi-axes a and b). The symmetry axis of the spheroid forms an angle θ with the A-axis of the parallelepiped. The center of the spheroid is fixed, but the angle θ can vary. Let $A \gg a$, $B \gg b$.



a) Calculate the electrostatic interaction energy U of this system to quadrupolar order. Show that U can be expressed in terms of e, the lattice constants A and B, and the quadrupole moment Q_{33} of the spheroid in the coordinate system of the lattice.

b) Calculate the quadrupole moment Q'_{33} of the spheroid in its principal-axes system, and then calculate Q_{33} by transforming into the lattice system. Express U as a function of the angle θ.

hint: In general, lining up the principal-axes systems would require three Euler angles. However, due to the symmetries of the problem Q'_{33} and Q_{33} in the present case are related by only one angle, viz., θ.

c) Find the equilibrium positions of the spheroid. Make sure to distinguish the cases of prolate and oblate spheroids ($a > b$ and $a < b$, respectively), as well as between the cases $A > B$ and $A < B$.

(15 points)

.../over

27. **Electric charges in an external field**

Consider a static electric charge distribution $\rho(\boldsymbol{x})$ subject to a static potential $\varphi(\boldsymbol{x})$. Consider the force $\boldsymbol{F}_{\rm el}$ on the charge distribution and show that $\boldsymbol{F}_{\rm el} = -\boldsymbol{\nabla} U$, with U the electrostatic energy calculated in ch.3 §3.6. In particular, convince yourself that the dipole term in the multipole expansion of U gives the correct potential energy for an electric dipole moment $\boldsymbol{d}$ in an electric field $\boldsymbol{E}$.

(3 points)

25.) $$\varphi(\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x}=0) + \vec{x}\cdot\vec{\nabla}\varphi\Big|_{\vec{x}=0} + \frac{1}{2} x_i x_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}\Big|_{\vec{x}=0} + \dots$$

$$=: \varphi_0 - \vec{x}\cdot\vec{E} + \frac{1}{2} x_i x_j \varphi_{ij} + \dots$$

$$\equiv \varphi_0 + \varphi_1(\vec{x}) + \varphi_2(\vec{x}) + \dots$$

a) $\rho(\vec{\xi}) = \rho(-\vec{\xi}) \rightsquigarrow \varphi(-\vec{x}) = \int d\vec{\xi}\, \frac{\rho(\vec{\xi})}{|\vec{x}+\vec{\xi}|} = \int d\vec{\xi}\, \frac{\rho(-\vec{\xi})}{|\vec{x}-\vec{\xi}|} = \varphi(\vec{x})$

1

$\rightsquigarrow$ All terms odd in $\vec{x}$ vanish, in particular $\underline{\vec{E} = 0}$

b) φ_{ij} is real symmetric $\rightsquigarrow \exists$ coordinate system such that φ_{ij} is diagonal

1

$\varphi(\vec{x})$ obeys Laplace's eq. $\forall\, |\vec{x}| < r_0$

$\rightsquigarrow \sum_i \varphi_{ii} = 0$

$$\rightsquigarrow \varphi_{ij} \text{ has the form } \varphi_{ij} = \begin{pmatrix} \varphi_+ + \varphi_- & 0 & 0 \\ 0 & \varphi_+ - \varphi_- & 0 \\ 0 & 0 & -2\varphi_+ \end{pmatrix}$$

1

where $\varphi_- = \frac{1}{2}(\varphi_{11} - \varphi_{22})$

$$\rightsquigarrow \underline{\varphi_2(\vec{x})} = \frac{1}{2} r^2 \sin^2\vartheta \cos^2\varphi\, (\varphi_+ + \varphi_-) + \frac{1}{2} r^2 \sin^2\vartheta \sin^2\varphi\, (\varphi_+ - \varphi_-) + \frac{1}{2} r^2 \cos^2\vartheta\, (-2\varphi_+)$$

1

$$= \underline{\frac{1}{2} r^2 \left[(1 - 3\cos^2\vartheta)\varphi_+ + \sin^2\vartheta \cos 2\varphi\, \varphi_- \right]}$$

Rotational invariance of $\rho(\vec{\xi})$ implies rotational invariance of $\varphi(\vec{x})$, and in particular of $\varphi_2(\vec{x})$

$$\rightsquigarrow \quad \underline{\varphi_2(r,\vartheta,\varphi+\alpha)} = \tfrac{1}{2} r^2 \left[(1-3\cos^2\vartheta)\,\varphi_+ + \sin^2\vartheta \cos 2(\varphi+\alpha)\,\varphi_- \right]$$

$$\stackrel{!}{=} \underline{\varphi_2(r,\vartheta,\varphi)}$$

1 $\quad \rightsquigarrow \quad \varphi_- \cos(2\varphi+2\alpha) = \varphi_- \cos 2\varphi \quad \rightsquigarrow \quad \underline{\underline{\varphi_- = 0}}$

c) Cubic symmetry $\rightsquigarrow$ $\rho(\vec{\xi})$ invariant under rotations through $\frac{\pi}{2}$ about any of the three axes x, y, z.

b) $\rightsquigarrow$ $\varphi_- = 0$ due to invariance under rotation about z-axis

Rotate about x or y $\rightsquigarrow$ $\vartheta \to \vartheta + \pi/2$

But invariance of $\rho(\vec{\xi})$ implies invariance of $\varphi(\vec{x})$

$$\rightsquigarrow \quad \underline{\varphi_2(r, \vartheta + \tfrac{\pi}{2}, \varphi)} = \tfrac{1}{2} r^2 \varphi_+ \left[1 - 3\cos^2(\vartheta + \pi/2) \right]$$

$$\stackrel{!}{=} \underline{\tfrac{1}{2} r^2 \varphi_+ \left[1 - 3\cos^2\vartheta \right]}$$

1 $\quad \rightsquigarrow \quad \varphi_+ \cos^2(\vartheta + \pi/2) = \varphi_+ \cos^2\vartheta \quad \rightsquigarrow \quad \underline{\varphi_+ = 0} \quad \rightsquigarrow \quad \underline{\underline{\varphi_{ij} = 0}}$

26.) a) cf J §3.6 → Consider the potential due to the 8 charges:

$$\varphi(\vec{x}) = e \sum_{\alpha=1}^{8} \frac{1}{|\vec{x}-\vec{\xi}^{(\alpha)}|} \quad \text{where} \quad \vec{\xi}^{(\alpha)} = \frac{1}{2}\begin{pmatrix} \pm\square \\ \pm\square \\ \pm A \end{pmatrix}$$

We need

(1) $$\underline{\varphi_0} = \varphi(\vec{x}=0) = e \sum_{\alpha=1}^{8} \frac{1}{|\vec{\xi}^{(\alpha)}|} = e\,\frac{8}{\sqrt{A^2/4+2\square^2/4}} = \underline{\frac{16e}{\sqrt{A^2+2\square^2}}}$$

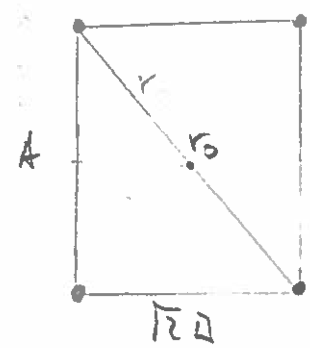
(1) $$\vec{E} = -\vec{\nabla}\varphi(\vec{x}=0) = \sum_{\alpha=1}^{8} \frac{-\vec{\xi}^{(\alpha)}}{|\vec{\xi}^{(\alpha)}|^3} = 0 \quad \text{by symmetry}$$

(1) $$\varphi_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}\varphi\Big|_{\vec{x}=0} = \begin{pmatrix} \varphi & 0 & 0 \\ 0 & \varphi & 0 \\ 0 & 0 & -2\varphi \end{pmatrix} \quad \text{by symmetry (see Problem 35e!)}$$

where

(1) $$\underline{\varphi} = \varphi_{11} = \frac{\partial^2}{\partial x^2}\Big|_{\vec{x}=0} e \sum_{\alpha=1}^{8} \frac{1}{|\vec{x}-\vec{\xi}^{(\alpha)}|} = e\left(\frac{3(\xi_1^{(\alpha)})^2}{|\vec{\xi}^{(\alpha)}|^5} - \frac{1}{|\vec{\xi}^{(\alpha)}|^3}\right)$$

Define $r_0 := \sqrt{A^2+2\square^2} \to |\vec{\xi}^{(\alpha)}| = \frac{1}{2}r_0$



$$\to \quad \underline{\varphi_0 = \frac{16e}{r_0}}$$

$$\underline{\varphi} = e\left(\frac{3\square^2/4}{(r_0/2)^5} - \frac{1}{(r_0/2)^3}\right)$$

(1) $$= e\,\frac{1}{r_0^5}\left(\frac{3}{\cancel{4}}\square^2\cdot\cancel{8}\cdot\cancel{4} - \cancel{8}r_0^2\right) = \frac{8e}{r_0^5}\left(3\square^2 - A^2 - 2\square^2\right) = \underline{\frac{8e}{r_0^5}\left(\square^2 - A^2\right)}$$

$$\to \quad \underline{\underline{U}} = \varphi_0 Q + \frac{1}{3}\left(\varphi Q_{11} + \varphi Q_{22} - 2\varphi Q_{33}\right)$$

$$= \varphi_0 Q + \frac{1}{3}\varphi\left(Q_{11} + Q_{22} - 2Q_{33}\right)$$

(1) $\sum_i Q_{ii} = 0 \;\leadsto$ $$= \underline{\underline{\varphi_0 Q - \varphi Q_{33}}}$$

remark: Here Q_{33} is the quadrupole moment of the spheroid in the lattice coordinate system!

b) In the principal-axes system of the spheroid the quadrupole tensor has the form
$$Q'_{ij} = \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & -2q \end{pmatrix}$$

Transform to the lattice system by means of rotation matrices (elements of $SO(3)$) D:

$$Q_{ij} = \sum_{n,m} D_{in} Q'_{nm} D_{jm}$$

$$\Rightarrow \underline{Q_{33}} = D_{31} Q'_{11} D_{31} + D_{32} Q'_{22} D_{32} + D_{33} Q'_{33} D_{33}$$
$$= q(D_{31})^2 + q(D_{32})^2 - 2q(D_{33})^2$$
$$= \underline{q\left[(D_{31})^2 + (D_{32})^2 - 2(D_{33})^2\right]}$$

Now D_{ij} is an orthogonal trafo $\Rightarrow$ $\underline{D_{31}^2 + D_{32}^2 + D_{33}^2 = 1}$

and D must align the z'-axis with the z-axis $\Rightarrow$ $\underline{D_{33} = \cos\theta}$

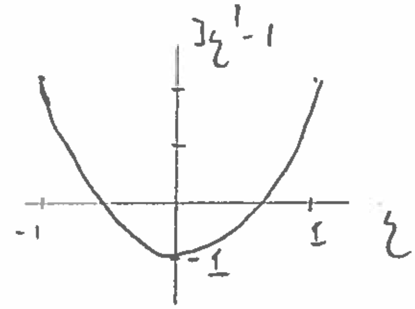
$$\Rightarrow \underline{Q_{33}} = q\left[1 - D_{33}^2 - 2D_{33}^2\right] = \underline{q\left[1 - 3\cos^2\theta\right]}$$

Finally, Problem 20 with $q = \frac{Q}{10}(b^2 - a^2)$

$$\Rightarrow \underline{\underline{U}} = \varphi_0 Q - \frac{8e}{r_0^5}(B^2 - A^2)\, q\,(1 - 3\cos^2\theta)$$
$$= \underline{\underline{\varphi_0 Q + \frac{8e}{r_0^5}\frac{Q}{10}(A^2 - B^2)(a^2 - b^2)(3\cos^2\theta - 1)}}$$

c) $3\zeta^2-1$ is minimal for $\zeta=0$

$\Leftrightarrow \vartheta=\pi/2$



maximal for $\zeta=\pm 1$

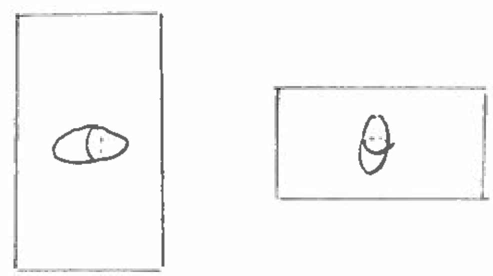
$\Leftrightarrow \vartheta = 0, \pi$

let $eQ>0$ $\rightarrow$ U is minimal for

$\vartheta = \frac{\pi}{2}$ if $(A^2-B^2)(a^2-b^2)>0$

1 $\vartheta = 0$ if $(A^2-B^2)(a^2-b^2)<0$

prolate spheroid $(a>b)$:
(cigar)



1

oblate $(a<b)$: flips the two cases
(disc)

1 $eQ<0$: Flips the two cases again.

27.) ch 2 §§ 2.5, 2.5c $\to$ The force on a single charge is $\vec{\nabla} e \varphi(\vec{x})$

$\to$ The force on $\rho(\vec{x}) = \sum_\alpha e_\alpha \delta(\vec{x} - \vec{x}_\alpha)$ is

(1)
$$\vec{F}_{el} = -\sum_\alpha e_\alpha \vec{\nabla}\varphi(\vec{x}) = -\int d\vec{x}\, \rho(\vec{x}) \vec{\nabla}\varphi(\vec{x}) \overset{\text{ch 2 §2.4}}{=} + \int d\vec{x}\, \rho(\vec{x}) \vec{E}(\vec{x})$$

Now expand as in ch 3 §3.6:

$$F_{el}^i = -\int d\vec{x}\, \rho(\vec{x}) \left[\partial_i \varphi \big|_{x=0} + x_j \partial_j \partial_i \varphi \big|_{x=0} + \dots \right]$$

$$= -Q\, \partial_i \varphi - (\partial_i \partial_j \varphi)_{\vec{x}=0} \int d\vec{x}\, x_j \rho(\vec{x}) + \dots \qquad \underline{Q = \int d\vec{x}\, \rho(\vec{x})}$$

$$\overset{\partial_j \varphi = -E_j}{=} -Q(\partial_i \varphi)_{\vec{x}=0} + (\partial_i E_j)_{\vec{x}=0}\, d_j + \dots \qquad \underline{\vec{d} = \int d\vec{x}\, \vec{x} \rho(\vec{x})}$$

$$= -\partial_i \Big|_{\vec{x}=0} \left[Q\varphi - \vec{E}\cdot\vec{d} + \dots \right]$$

(1)
$\to$ $\underline{\underline{\vec{F}_{el} = -(\vec{\nabla} U)_{\vec{x}=0}}}$ with $U(\vec{x}) = Q\varphi(\vec{x}) - \vec{d}\cdot\vec{E}(\vec{x})$ from ch 3 §3.6 and $\vec{x}$ is put to zero after calculating the gradient.

In particular, the dipole term in the electrostatic energy,

$$\underline{\underline{U_{dipole} = -\vec{E}\cdot\vec{d}}}$$

(1)
which describes the potential energy of a fixed electric dipole $\vec{d}$ in an electric field $\vec{E}$.