PHYS 622 | Classical Electrodynamics | W 2019

Problem Assignment # 8 | 02/27/2019
due 03/06/2019

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33. Wave equations for the electromagnetic fields

Show directly from the Maxwell equations, without introducing potentials, that the fields obey the inhomogeneous wave equations

$$\Box \boldsymbol{E} = -4\pi \left(\boldsymbol{\nabla}\rho + \frac{1}{c^2}\partial_t \boldsymbol{j} \right) \qquad , \qquad \Box \boldsymbol{B} = \frac{4\pi}{c} \boldsymbol{\nabla} \times \boldsymbol{j} \ .$$

(2 points)

34. Polaritons

As a model for a dielectric, consider a polarization field $\boldsymbol{P}(\boldsymbol{x}, t)$ that determines the sources of the electromagnetic fields according to

$$\boldsymbol{j} = \partial_t \boldsymbol{P} \quad , \quad \rho = -\boldsymbol{\nabla} \cdot \boldsymbol{P} \quad .$$

In addition to Maxwell's equations, the dynamics of the system are governed by an equation of motion for $\boldsymbol{P}$,

$$\left(\partial_t^2 + \omega_0^2\right) \boldsymbol{P}(\boldsymbol{x}, t) = a^2\, \boldsymbol{E}(\boldsymbol{x}, t) \quad (*) \quad ,$$

where ω_0 is a characteristic frequency and a is a real parameter (which dimensionally also is a frequency). This models the dielectric as a harmonic oscillator that is driven by the electric field.

a) Show that Maxwell's equations plus $(*)$ have solutions given by both longitudinal ($\boldsymbol{k} \parallel \boldsymbol{E}, \boldsymbol{P}$) and transverse ($\boldsymbol{k} \perp \boldsymbol{E}, \boldsymbol{P}$) monochromatic plane waves, and find the frequency-wavenumber relations for the various solutions.

b) Show that the transverse waves in the long-wavelength limit are photon-like, viz., $\omega_T(\boldsymbol{k} \to 0) = (c/n)|\boldsymbol{k}|$, and determine the index of refraction n.

c) Show that no homogeneous wave propagation is possible in a frequency band $\omega_- < \omega < \omega_+$, and find $\omega_\mp$. Derive the Lyddane-Sachs-Teller relation

$$\omega_+^2 / \omega_-^2 = \epsilon(\omega = 0)$$

where $\epsilon(\omega) = 1 + 4\pi a^2/(\omega_0^2 - \omega^2)$ is the dielectric function of the dielectric.

d) Discuss the frequency-wavenumber relation for all possible waves explicitly, especially in the limits $k \to 0$ and $k \to \infty$, and plot the result.

(14 points)

.../over

35. **Liénard-Wiechert potentials**

Consider a point charge e that moves on a given trajectory $\boldsymbol{X}(t)$ with velocity $\boldsymbol{v}(t) = \dot{\boldsymbol{X}}(t)$ which results in charge and current densities

$$\rho(\boldsymbol{x}, t) = e\,\delta(\boldsymbol{x} - \boldsymbol{X}(t)) \qquad , \qquad \boldsymbol{j}(\boldsymbol{x}, t) = e\,\boldsymbol{v}(t)\,\delta(\boldsymbol{x} - \boldsymbol{X}(t))$$

Show that the resulting retarded potentials have the form

$$\varphi(\boldsymbol{x}, t) = \frac{e}{|\boldsymbol{x} - \boldsymbol{X}(t_-)| - \boldsymbol{v}(t_-) \cdot (\boldsymbol{x} - \boldsymbol{X}(t_-))/c}$$

$$\boldsymbol{A}(\boldsymbol{x}, t) = \frac{1}{c}\,\boldsymbol{v}(t_-)\,\varphi(\boldsymbol{x}, t)$$

where t_- is the solution of

$$t_- = t - \frac{1}{c}\,|\boldsymbol{x} - \boldsymbol{X}(t_-)| \qquad (*)$$

These are known as Liénard-Wiechert potentials after Alfred-Marie Liénard and Emil Wiechert, who derived them in 1898 and 1900, respectively.

hint: Show that the equation $(*)$ for t_- has one and only one solution.

(6 points)

36. **Potential of a uniformly moving charge**

Consider a charge e moving uniformly along the x-axis with velocity v: $\boldsymbol{X}(t) = (vt, 0, 0)$. Determine the Liénard-Wiechert potentials explicitly, and show that the result is that same as the one obtained in ch. 2 §2.4 by means of a Lorentz transformation.

(6 points)

33.) $\underline{\underline{\Box \vec{B}}} = \left(\frac{1}{c^2}\partial_t^2 - \vec{\nabla}^2\right)\vec{B} =$

$$= \frac{1}{c}\partial_t\left(\frac{1}{c}\partial_t \vec{B}\right) + \vec{\nabla}\times\vec{\nabla}\times\vec{B} \qquad \text{via } \vec{\nabla}\times\vec{\nabla}\times\vec{v} = \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) - \vec{\nabla}^2\vec{v} \text{ and } \vec{\nabla}\cdot\vec{B} = 0$$

$$= \cancel{\frac{1}{c}\partial_t\left(-\vec{\nabla}\times\vec{E}\right)} + \frac{4\pi}{c}\vec{\nabla}\times\vec{j} + \cancel{\frac{1}{c}\partial_t\vec{\nabla}\times\vec{E}} \qquad \text{by M-eqs (2)+(4)}$$

(1) $$= \underline{\underline{\frac{4\pi}{c}\vec{\nabla}\times\vec{j}}}$$

$$\underline{\frac{1}{c^2}\partial_t^2\vec{E}} = \frac{1}{c}\partial_t\left(-\frac{4\pi}{c}\vec{j} + \vec{\nabla}\times\vec{B}\right) \qquad \text{by M-eq. (4)}$$

$$= -\frac{4\pi}{c^2}\partial_t\vec{j} + \vec{\nabla}\times\left(-\vec{\nabla}\times\vec{E}\right) \qquad \text{by M-eq. (2)}$$

$$= -\frac{4\pi}{c^2}\partial_t\vec{j} - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) + \vec{\nabla}^2\vec{E}$$

$$= -\frac{4\pi}{c^2}\partial_t\vec{j} - 4\pi\vec{\nabla}\rho + \vec{\nabla}^2\vec{E} \qquad \text{by M-eq. (3)}$$

(1) $$\rightarrow \underline{\underline{\Box\vec{E} = -4\pi\left(\vec{\nabla}\rho + \frac{1}{c^2}\partial_t\vec{j}\right)}}$$

34.) a) Maxwell eqs.: $\vec\nabla\cdot\vec E = 4\pi\rho = -4\pi\vec\nabla\cdot\vec P \rightarrow \underline{\vec\nabla\cdot(\vec E+4\pi\vec P)=0}$ (1)

$\underline{\vec\nabla\times\vec E = -\frac{1}{c}\partial_t\vec B}$ (2)

$\underline{\vec\nabla\cdot\vec B = 0}$ (3)

$\vec\nabla\times\vec B = \frac{4\pi}{c}\vec j + \frac{1}{c}\partial_t\vec E = \frac{4\pi}{c}\partial_t\vec P + \frac{1}{c}\partial_t\vec E \rightarrow \underline{\vec\nabla\times\vec B = \frac{1}{c}\partial_t(\vec E+4\pi\vec P}$ (4)

Eq. of motion: $(\partial_t^2+\omega_0^2)\vec P = e^2\vec E$ (*)

Look for monochromatic plane-wave solutions:

$$\vec E(\vec x,t) = \int\frac{d k}{(2\pi)^3}\, e^{i(\vec k\vec x-\omega t)}\vec E_{\vec k} \quad \text{etc.}$$

1

(*) $\rightarrow (\omega_0^2-\omega^2)\vec P_{\vec k} = e^2\vec E_{\vec k}$

$\rightarrow \underline{P_{\vec k} = \chi(\omega)E_{\vec k}}$ with $\underline{\chi(\omega) = \frac{e^2}{\omega_0^2-\omega^2}}$

1

(1) $\rightarrow \underline{0} = \vec k\cdot(\vec E_{\vec k}+4\pi\vec P_{\vec k}) = \vec k\cdot(1+4\pi\chi(\omega))\vec E_{\vec k} = \underline{\varepsilon(\omega)\,\vec k\cdot\vec E_{\vec k}}$

with $\underline{\varepsilon(\omega) = 1+\frac{4\pi e^2}{\omega_0^2-\omega^2}}$

(2) $\rightarrow \underline{\vec k\times\vec E_{\vec k} = \frac{\omega}{c}\vec B_{\vec k}}$

(3) $\rightarrow \underline{\vec k\cdot\vec B_{\vec k} = 0}$

1

(4) $\rightarrow \underline{\vec k\times\vec B_{\vec k} = -\frac{\omega}{c}\varepsilon(\omega)\vec E_{\vec k}}$

longitudinal waves: $\underline{\vec k \parallel \vec E_{\vec k}, \vec P_{\vec k}}$

(1) $\rightarrow \varepsilon(\omega)=0 = 1+\frac{4\pi e^2}{\omega_0^2-\omega^2} \rightarrow \omega^2 = \omega_0^2+4\pi e^2$

1

$\rightarrow \underline{\omega = \omega_0\sqrt{1+4\pi e^2/\omega_0^2} = \omega_0\,\varepsilon(\omega=0) =: \omega}$

(2) -> $\underline{\vec{B}_{\vec{k}}^{\,0}=0}$ since $\omega=\omega_+\neq 0$ and $\vec{k}\times\vec{E}_{\vec{k}}=0$

(3), (4) are now automatically fulfilled.

-> A purely electric plane wave will propagate ω_+ is a solution for any $\vec{k}$

1

Transverse waves: $\vec{k}\perp\vec{E},\vec{P}$

(1) always fulfilled

(2) + (4) -> $\vec{k}\times(\vec{k}\times\vec{E}_{\vec{k}}) = \frac{\omega}{c}\,\vec{k}\times\vec{B}_{\vec{k}} = -\frac{\omega^2}{c^2}\,\varepsilon(\omega)\,\vec{E}_{\vec{k}}$

$k^2\vec{E}_{\vec{k}}$ -> $k^2 = \frac{\omega^2}{c^2}\,\varepsilon(\omega) = \frac{\omega^2}{c^2}\left(1+\frac{4\pi e^2}{\omega_0^2-\omega^2}\right)$

1 (2) -> $\vec{k}\cdot\vec{B}_{\vec{k}} = \frac{c}{\omega}\,\vec{k}\cdot(\vec{k}\times\vec{E}_{\vec{k}}) = 0$ -> (3) is fulfilled

-> Transverse plane-wave solutions exist with $\omega(k)$ given as the solution of $\boxed{\omega^2\varepsilon(\omega) = c^2k^2}$ (+)

1

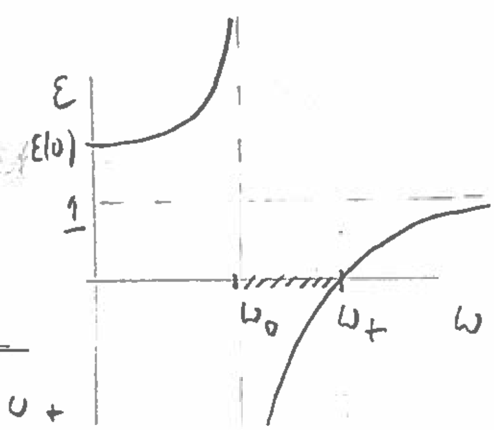
b) $\omega\to 0$ -> $\varepsilon(\omega)\to\varepsilon(\omega=0) = 1+4\pi e^2/\omega_0^2 =: n^2$

-> $\omega(k\to 0) = \frac{c}{n}\,k$

1

c) (+) -> Propagating solution only for $\varepsilon(\omega)>0$. But

$\varepsilon(\omega)<0$ for $\omega_-\equiv\omega_0<\omega<\omega_+$



1 -> No propagating solution for $\omega_-<\omega<\omega_+$

$\varepsilon(\omega)$ can be written

$$\varepsilon(\omega) = \frac{\omega_0^2 + 4\pi c^2 - \omega^2}{\omega_0^2 - \omega^2} = \frac{\omega_+^2 - \omega^2}{\omega_-^2 - \omega^2}$$

$\rightarrow$ $\omega_+^2/\omega_-^2 = \varepsilon(\omega = 0)$ Lyddane-Sachs-Teller

d) Part a) (+) $\rightarrow$ $\omega^2 \frac{\omega_+^2 - \omega^2}{\omega_-^2 - \omega^2} = c^2 k^2$

$\rightarrow$ $-\omega^4 + \omega_+^2\omega^2 = -c^2k^2\omega^2 + \omega_-^2 c^2 k^2$

$\rightarrow$ $\omega^4 - (\omega_+^2 + c^2k^2)\omega^2 + \omega_-^2 c^2 k^2 = 0$

$\rightarrow$ $\omega_{1,2}^2 = \frac{1}{2}\left[\omega_+^2 + c^2k^2 \pm \sqrt{(\omega_+^2 + c^2k^2)^2 - 4\omega_-^2 c^2 k^2}\right]$

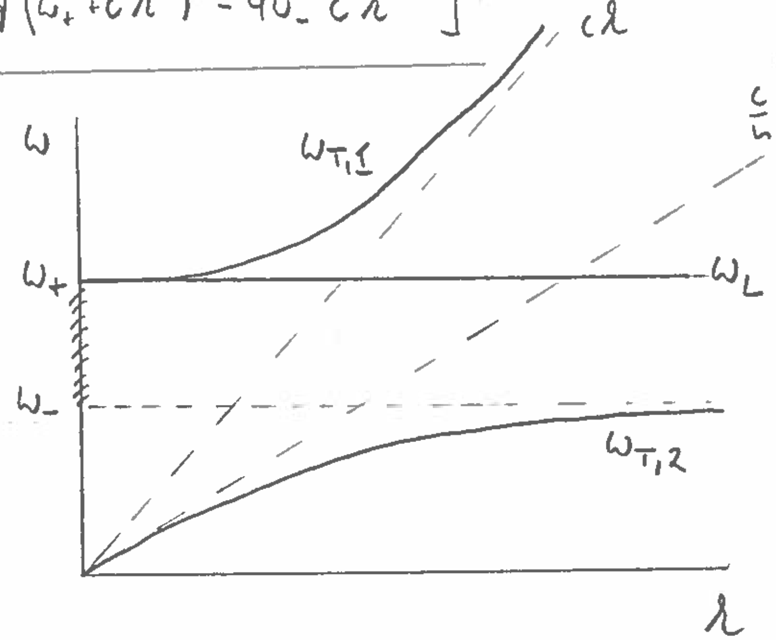
$\omega_1(k \to 0) = \omega_+ + O(k^2)$

$\omega_1(k \to \infty) = ck - O(1/k)$

$\omega_2(k \to 0) = \frac{c}{n}k - O(k^3)$

$\omega_2(k \to \infty) = \omega_- - O(1/k^2)$

Then are the two transverse branches.



In addition, there is the longitudinal branch with

$\omega(k) \equiv \omega_+$

35.) use §2.3 ~>

$$\underline{\underline{\varphi(\vec{x},t)}} = \int d\vec{x}'dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t-t'-\tfrac{1}{c}|\vec{x}-\vec{x}'|\right) \rho(\vec{x}',t')$$

$$= \int d\vec{x}'dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t-t'-\tfrac{1}{c}|\vec{x}-\vec{x}'|\right) e\, \delta(\vec{x}'-\vec{X}(t'))$$

$$\underline{= e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \delta\left(t'-t+\tfrac{1}{c}|\vec{x}-\vec{X}(t')|\right)}$$

define $f(t') = t'-t+\frac{1}{c}|\vec{x}-\vec{X}(t')|$

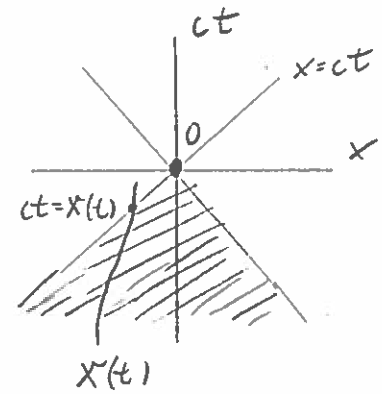
~> Need the zeros of $f(t')$.

1

let $(ct,\vec{x}) = (0,0)$ wlog.

~> $ct' = |\vec{X}(t')|$

solution = intersection between world line of particle with the light cone



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speed of particle < c ~> there is one and only one intersection point

~> $\delta(f(t')) = \frac{1}{|f'(t_-)|}\delta(t'-t_-)$

when $f(t_-)=0 \iff \boxed{t_- = t - \frac{1}{c}|\vec{x}-\vec{X}(t_-)|}$ (*)

1

Discussion above ~> (*) has a unique solution

and

$$\underline{f'(t')} = \frac{d}{dt'}f(t') = 1 + \frac{1}{c}\frac{1}{|\vec{x}-\vec{X}(t')|}(-)\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))$$

$$\underline{= 1 - \frac{1}{c}\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))\frac{1}{|\vec{x}-\vec{X}(t')|}} > 0 \text{ since } \frac{|\vec{v}|}{c} < 1$$

1

$$\rightarrow \underline{\underline{\varphi(\vec{x},t)}} = e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \frac{1}{1-\frac{1}{c}\vec{v}(t')\cdot(\vec{x}-\vec{X}(t'))\frac{1}{|\vec{x}-\vec{X}(t')|}} \delta(t'-t_-)$$

$$= \underline{\underline{\frac{e}{|\vec{x}-\vec{X}(t_-)| - \frac{1}{c}\vec{v}(t_-)\cdot(\vec{x}-\vec{X}(t_-))}}}$$

1

For $\vec{A}$ we have

$$\vec{A}(\vec{x},t) = \frac{1}{c}\int d\vec{x}'dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta\left(t-t'-\frac{1}{c}|\vec{x}-\vec{x}'|\right) \vec{j}(\vec{x}',t')$$

$$= \frac{e}{c}\int d\vec{x}'dt' \frac{1}{|\vec{x}-\vec{x}'|} \delta(f(t'))\, \vec{v}(t')\, \delta(\vec{x}'-\vec{X}(t'))$$

$$= \frac{e}{c}\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \vec{v}(t') \frac{1}{|f'(t_-)|} \delta(t'-t_-)$$

$$= \frac{1}{c}\vec{v}(t_-)\, e\int dt' \frac{1}{|\vec{x}-\vec{X}(t')|} \frac{1}{|f'(t_-)|} \delta(t'-t_-)$$

$$= \underline{\underline{\frac{1}{c}\vec{v}(t_-)\,\varphi(\vec{x},t)}}$$

1

36.) Consider Problem 35 for the special case

$$\vec{x}(t) = (vt, 0, 0)\ , \quad \vec{v}(t) = (v, 0, 0)$$

→ The eq. for t_- reads

(1) $$t_- = t - \frac{1}{c}\sqrt{(x-vt_-)^2 + y^2 + z^2}$$

→ $$x - vt_- = x - vt - \frac{v}{c}\sqrt{(x-vt_-)^2 + y^2 + z^2}$$

→ $$(x-vt_-)^2 - 2(x-vt)(x-vt_-) + (x-vt)^2 = \frac{v^2}{c^2}(x-vt_-)^2 + \frac{v^2}{c^2}(y^2+z^2)$$

→ $$(x-vt_-)^2\gamma^{-2} - 2(x-vt)(x-vt_-) + (x-vt)^2 - \frac{v^2}{c^2}(y^2+z^2) = 0$$

$\gamma = \frac{1}{\sqrt{1-v^2/c^2}}$ → $$x - vt_- = \frac{1}{\cancel{2}}\gamma^2\left[\cancel{2}(x-vt) \pm \sqrt{\cancel{4}(x-vt)^2 - \cancel{4}\gamma^{-2}(x-vt)^2 + \cancel{4}\gamma^{-2}\frac{v^2}{c^2}(y^2+z^2)}\right]$$

$$= \gamma^2\left[x - vt \pm \sqrt{\frac{v^2}{c^2}(x-vt)^2 + \frac{v^2}{c^2}(y^2+z^2)\gamma^{-2}}\right]$$

(1) $$= \gamma^2(x-vt) \overset{+}{(-)} \frac{v}{c}\gamma^2\sqrt{(x-vt)^2 + \gamma^{-2}(y^2+z^2)}$$

The physical (= retarded) solution yields the smaller value for t_- → The physical solution has "+" in the above eq.

Define $R^*(\vec{x},t) := \sqrt{(x-vt)^2 + \gamma^{-2}(y^2+z^2)}$ as in cl J § 2.4

(1) → $$x - vt_- = \gamma^2\left[x - vt + \frac{v}{c}R^*(\vec{x},t)\right]$$ This is the explicit solution for t_-.

Problem 35 →

(1) $$\frac{e}{\varphi(\vec{x},t)} = \sqrt{(x-vt_-)^2 + y^2 + z^2} - \frac{v}{c}(x - vt_-)$$

cl J § 2.4 → I need to show that the rhs equals R^*.

$$\to \sqrt{(x-vt_-)^2+y^2+z^2} - \frac{v}{c}(x-vt_-) \overset{?}{=} R^*$$

$$\to (x-vt_-)^2+y^2+z^2 \overset{?}{=} \left(R^*+\frac{v}{c}(x-vt_-)\right)^2$$
$$= \frac{v^2}{c^2}(x-vt_-)^2 + 2\frac{v}{c}(x-vt_-)R^* + (R^*)^2$$

$$\to (x-vt_-)^2\left(1-\frac{v^2}{c^2}\right) + y^2+z^2 \overset{?}{=} (R^*)^2 + 2\frac{v}{c}(x-vt_-)R^*$$

$$\to \gamma^2\left[(x-vt)+\frac{v}{c}R^*\right]^2 + y^2+z^2 \overset{?}{=} (R^*)^2 + 2\frac{v}{c}\gamma^2\left[(x-vt)+\frac{v}{c}R^*\right]R^*$$

$$\to \gamma^2(x-vt)^2 + \cancel{2\gamma^2\frac{v}{c}(x-vt)R^*} + \cancel{\gamma^2\frac{v^2}{c^2}(R^*)^2} + y^2+z^2 \overset{?}{=} (R^*)^2 + \cancel{2\frac{v}{c}\gamma^2(x-vt)R^*} + \cancel{2}\frac{v^2}{c^2}\gamma^2(R^*)^2$$

$$\to \gamma^2(x-vt)^2+y^2+z^2 \overset{?}{=} (R^*)^2\left(1+\frac{v^2}{c^2}\gamma^2\right)$$
$$= (R^*)^2\left(1+\frac{v^2/c^2}{1-v^2/c^2}\right) = (R^*)^2\frac{1}{1-v^2/c^2} = \gamma^2(R^*)^2$$

(1) $$\to (R^*)^2 \overset{!}{=} (x-vt)^2 + \gamma^{-2}(y^2+z^2)$$ ✓

Now I have

$$\frac{e}{\varphi(\vec{x},t)} = \sqrt{(x-vt_-)^2+y^2+z^2} - \frac{v}{c}(x-vt_-) = R^*(\vec{x},t)$$

$$\to \varphi(\vec{x},t) = \frac{e}{R^*(\vec{x},t)}$$

and from Problem 4c,

$$\vec{A}(\vec{x},t) = \frac{\vec{v}}{c}\varphi(\vec{x},t) = \frac{e\vec{v}}{cR^*(\vec{x},t)}$$

(1) Then we indeed the same results as in ch 3 § 2.4

remark: solution via Lorentz transfo, as in ch 3 § 2.4, is much easier!