Midterm Exam   10/22/2020
due 11/05/2020

This is a take-home exam that runs concurrently with the homework. It is due by 5pm on Thursday, 11/5/2020 by email to dbelitz@you-know-where. You can consult any inanimate resource you like, but do not get help from live resources.

If you encounter logical gaps in your proofs that you can't fill, state clearly what you are assuming to be true but could not prove, and continue.

Credit breakdown: 2 points for Lemma 1, 3 points for Lemma 2, 6 points for Lemma 3, 2 points for the Theorem, for a total of 13 points.

Let X and Y be sets. Prove the following statements. (It is advisable to prove them in the order in which they are stated.)

Lemma 1: Let $f : X \to Y$ and $g : Y \to X$ be mappings such that $f \circ g = \mathrm{id}_Y$. Then f is surjective and g is injective.

Lemma 2: Let $f : X \to Y$ be surjective. Then there exists an injective mapping $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Lemma 3: Let $f : X \to Y$ be bijective. Then there exists a unique bijective mapping $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

Theorem: If $f : X \to Y$ is bijective, then there exists a unique mapping $f^{-1} : Y \to X$ that is also bijective and satisfies

(i) $f \circ f^{-1} = \mathrm{id}_Y$

(ii) $f^{-1} \circ f = \mathrm{id}_X$

(iii) $\left(f^{-1}\right)^{-1} = f$

Note: f^{-1} is the inverse of f whose existence we asserted, but did not prove, in §1.2 remark (3).