§3 Fields

3.1 Bilinear mappings

Def. 1: Let A, B, C be addition groups with neutral elements $0_A, 0_B, 0_C$, respectively. Let $\varphi : A \times B \to C$ be a mapping $\varphi(a,b) \equiv a \cdot b \equiv ab \in C$ and that, $\forall\, a, a_1, a_2 \in A$ and $b, b_1, b_2 \in B$, the distribution laws hold, i.e.

(i) $(a_1 + a_2) \cdot b = a_1 \cdot b + a_2 \cdot b$

(ii) $a_1 \cdot (b_1 + b_2) = a_1 b_1 + a_1 b_2$

Then φ is called bilinear.

Remark: (1) In (i), "+" on the lhs is the addition on A. In (ii), "+" on the lhs is the addition on B. In both cases, "+" on the rhs is the addition on C.

(2) "$\cdot$" is usually called multiplication, and referred to as an exterior operation, as opposed to the "interior" addition.

Properties: (1) $0_A \cdot b = a \cdot 0_B = 0_C \quad \forall\, a \in A, b \in B$.

(2) $(-a) \cdot b = a \cdot (-b) = -(a \cdot b) \quad \forall\, a \in A, b \in B$.

(3) $(-a) \cdot (-b) = a \cdot b \quad \forall\, a \in A, b \in B$.

Proof: (1) $0_A = 0_A + 0_A \rightsquigarrow 0_A \cdot b = (0_A + 0_A) \cdot b = 0_A \cdot b + 0_A \cdot b$

$\rightsquigarrow \underline{0_C} = 0_A \cdot b - 0_A \cdot b = 0_A \cdot b + 0_A \cdot b - 0_A \cdot b = \underline{0_A \cdot b}$

$0_B = 0_B + 0_B \rightsquigarrow a \cdot 0_B = a \cdot 0_B + a \cdot 0_B \rightsquigarrow 0_C = a \cdot 0_B$

(2) (1) $\to$ $0_C = 0_A \cdot b = (-a+a)\cdot b = (-a)\cdot b + a\cdot b$

$$0_C = a\cdot 0_B = a\cdot(-b+b) = a\cdot(-b) + a\cdot b$$

But 0_C is unique $\to$ $\underline{-(a\cdot b) = (-a)\cdot b = a\cdot(-b)}$

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(3) (2) $\to$ $\underline{(-a)\cdot(-b)} = -(a\cdot(-b)) = -(-(a\cdot b)) \overset{-(-c)=c\ \forall c\in C}{=} \underline{a\cdot b}$ $\square$

3.2 Fields

<u>def. 1</u>: Let $(K,+)$ be an additive group (with neutral element 0). In addition, let $\cdot : K\times K \to K$ be an associative bilinear multiplication. If $K\setminus\{0\}$ is a group under $\cdot$, then K is called a <u>field</u>.

<u>example</u>: (1) $\mathbb{R}$ under ordinary addition and multiplication is a commutative field. So is $\mathbb{Q}$. $\mathbb{Z}$ is not (e.g., 2 has no inverse under multiplication).

Problem 12
$\mathbb{Q}$ is a field
complex field example

3.3 The field $\mathbb{C}$ of complex numbers

<u>theorem</u>: We can construct a commutative field $\mathbb{C}$, called <u>complex numbers</u> with the following properties:

(1) $\mathbb{R} \subset \mathbb{C}$

(2) The number -1 is the square of an element $i \in \mathbb{C}$

(3) Each $z\in\mathbb{C}$ can be uniquely written $z = z_1 + iz_2$ $\equiv z' + iz''$ with $z_{1,2}\in\mathbb{R}$

<u>remark</u>: (1) z_1 and z_2 are called <u>real part</u> and <u>imaginary part</u> of z, respectively. $z_1 - iz_2 =: z^*$ (sometimes with $\bar{z}$) is called <u>complex conjugate</u> of z.

(2) We will denote the real (imaginary) part of z sometimes

proof: (i) Consider $\mathbb{R}\times\mathbb{R}$. Let $a=(a_1,a_2)$, $b=(b_1,b_2)\in\mathbb{R}\times\mathbb{R}$.

Define an addition "+" on $\mathbb{R}\times\mathbb{R}$ by

$$a+b := (a_1+b_1,\, a_2+b_2)$$

Then $\mathbb{R}\times\mathbb{R}$ is an additive group with neutral element $(0,0)$.

(ii) Define a multiplication on $\mathbb{R}\times\mathbb{R}$ by

$$ab := (a_1b_1-a_2b_2,\, a_1b_2+b_1a_2)$$

with a_1b_1 etc the ordinary multiplication in $\mathbb{R}$. Then

$$\underline{ba} = (b_1a_1-b_2a_2,\, b_1a_2+a_1b_2) = \underline{ab}$$

and

$\underline{a(b+b')} = \underline{ab+ab'}$ by direct calculation

(iii) $a(bc)=(ab)c$ by direct calculation

(iv) Let $a=(a_1,a_2)\neq(0,0) \rightarrow a_1^2+a_2^2>0$

$$\rightarrow (a_1,a_2)\left(\frac{a_1}{a_1^2+a_2^2},\, \frac{-a_2}{(a_1^2+a_2^2)}\right) = (1,0)$$

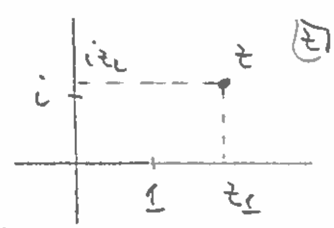
(v) $(0,1)^2 = (0,1)(0,1) = (-1,0)$

We now define $\mathbb{C}$ by means of an isomorphism $\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{C}$

that maps. $(0,1)\mapsto i\in\mathbb{C}$

$(z_1,z_2)\mapsto z\in\mathbb{C} \rightarrow z=z_1+iz_2$ □

remark: (2) The isomorphism is graphically represented by the "complex plane $\boxed{z}$", which is isomorphic to $\mathbb{R}_2$.

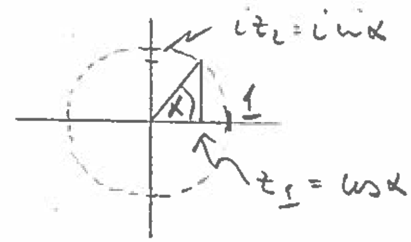


proposition. The set of complex numbers $\{e^{i\alpha};\, 0\leq\alpha<2\pi\}$ forms a circle of radius $\underline{1}$ and origin $0+i0$ in the complex plane, and proof: e^{anything} is defined by the power series $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$

$$\Rightarrow \underline{e^{i\alpha}} = 1 + i\alpha + \tfrac{1}{2}(i\alpha)^2 + \tfrac{1}{3!}(i\alpha)^3 + \ldots$$
$$= 1 + i\alpha - \tfrac{1}{2}\alpha^2 - \tfrac{i}{3!}\alpha^3 + \ldots$$
$$= \left(1 - \tfrac{1}{2}\alpha^2 + \ldots\right) + i\left(\alpha - \tfrac{1}{3!}\alpha^3 + \ldots\right) = \underline{\cos\alpha + i\sin\alpha}$$

$\Rightarrow z_1^2 + z_2^2 = \cos^2\alpha + \sin^2\alpha = \underline{1}$

i.e. circle of radius $\underline{1}$

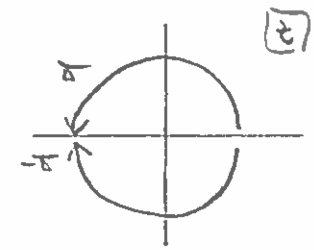


corollary: Let $z \in \mathbb{C}$. Then $\exists\, r \in [0, \infty[\, \subset \mathbb{R}$, $\varphi \in [-\pi, \pi[\, \subset \mathbb{R}$:

$$\boxed{z = re^{i\varphi}}$$

proof: proposition $\Rightarrow$ $z = z_1 + iz_2 = r\cos\varphi + ir\sin\varphi$ $\Big\}$ $\Rightarrow$ $\underline{r = \sqrt{z_1^2 + z_2^2}}$, $\underline{\varphi = \arctan(z_2/z_1)}$

remark: (3) r is called modulus of z and often denoted by $|z|$. φ is called argument of z.



(4) $-\pi \le \varphi < \pi$ is just a particular convention. φ can be defined on any other interval of length 2π.

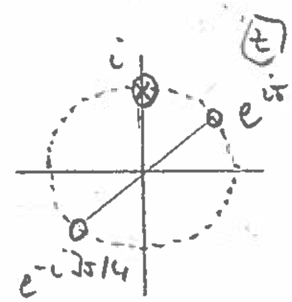
(5) An equivalent statement is $r \geq 0$, $\varphi \in \mathbb{R} \pmod{2\pi}$: $e^{i2\pi n} = 1 \;\forall\, n \in \mathbb{Z} \Rightarrow (r, \varphi)$ and $(r, \varphi + 2\pi n)$ represent the same $z \in \mathbb{C}$.

def.: Let $z \in \mathbb{C}$. Define real powers of z by

$$\boxed{z^x := r^x e^{i\varphi x}} \qquad x \in \mathbb{R}$$

remark: (6) This is consistent with §2.2 corollary (b).

(7) z^x is not unique for $x \notin \mathbb{N}$. In particular, $x = 1/n$ with $n \in \mathbb{N}$ yields n different values called roots.



example: $z = i = 1 \times e^{i\pi/2} \equiv e^{i(\frac{\pi}{2}+2\pi)}$

$i^{1/2} = e^{i\pi/4}$ or $e^{i(\frac{\pi}{4}+\pi)} = e^{i5\pi/4} \equiv e^{-i3\pi/4}$

§4 Vector spaces and linear spaces

4.1 Vector spaces

def. 1: Let $(V,+)$ be an additive group with neutral element 0, and let K be a field. Let there be an exterior multiplication $\varphi: K \times V \to V$ that is

(i) bilinear

(ii) associative in the sense $(\lambda\mu)x = \lambda(\mu x)$ $\forall x \in V$, $\lambda, \mu \in K$

(iii) obeys $1_K x = x$ $\forall x \in V$, where 1_K is the multiplicative neutral element of K.

Then we call V a vector space or linear space over K, or a K-vector space.

remark: (1) The elements of V are called vectors; the elements of K are often referred to as scalars.