proposition: Let Δ be a normal coordinate transfo. Then $\boxed{\det \Delta = \pm 1}$

proof: def. 1 $\Rightarrow$ $\det g = \det(\Delta^T g \Delta) \overset{\S 4.8.2 \text{ prop } 1}{=} \det g \, (\det \Delta)^2$

$\Rightarrow (\det \Delta)^2 = 1 \Rightarrow \det \Delta = \pm 1$ $\square$

§5 Tensor fields

5.1 Tensor fields

Let V be $\mathbb{R}_n$ endowed with a (generalized) metric g as defined in §4.8.1. Let Δ be a normal coordinate transfo from a normal coordinate system CS to a normal coordinate system $\widetilde{CS}$:

$$\tilde{x}^i = \Delta^i{}_j x^j$$

def. 1: For any $x \in V$, consider a rank-N tensor $t^{i_1 \dots i_N}(x)$.
We call $t^{i_1 \dots i_N}(x)$ a tensor field if, under a normal coordinate transfo,

$$\boxed{\tilde{t}^{i_1 \dots i_N}(\tilde{x}) = \Delta^{i_1}{}_{j_1} \dots \Delta^{i_N}{}_{j_N} \, t^{j_1 \dots j_N}(x)} \quad (*)$$

remark: (1) This generalizes the tensor concept of §4.3 by assigning a different tensor to every vector in V.
(2) Tensor fields are tensor-valued functions on V.

proposition: For the special case of homogeneous tensor fields, i.e., if $t^{i_1 \dots i_N}(x) \equiv t^{i_1 \dots i_N}$ independent of x, we recover the usual tensor concept of §4.3

proof: §4.3 prop. + remark (2) $\Rightarrow$

$$f(x,y,\dots) = t_{ij\dots}\, x^i y^j \dots = t^{ij\dots}\, x_i y_j \dots$$

when $f: V\times V\times\dots \to \mathbb{R}$ is the multilinear form that defines the tensor t. Now basis sets $\{e_i\}, \{\tilde e_i\}$ and expand the vector

$$x = x^i e_i = \tilde x^i \tilde e_i$$

when $\tilde x^i = \Lambda^i{}_j x^j$, $\tilde e_i = (\Lambda^{-1})^j{}_i e_j$ will be coordinate transfo
see §4.8.2

$$\Rightarrow f(x,y,\dots) = f(x_i e^i, y_j e^j, \dots) = x_i y_j \dots \overbrace{f(e^i, e^j, \dots)}^{= t^{ij\dots}} = t^{ij\dots} x_i y_j \dots$$

$$= f(\tilde x_i \tilde e^i, \tilde y_j \tilde e^j \dots) = \tilde x_i \tilde y_j \dots \underbrace{f(\tilde e^i, \tilde e^j, \dots)}_{= \tilde t^{ij\dots}} = \tilde t^{ij\dots} \tilde x_i \tilde y_j \dots$$

§4.8.2 remark (2) $\Rightarrow$ $x_i = g_{ij} x^j = g_{ij} (\Lambda^{-1})^j{}_l \tilde x^l = (g\Lambda^{-1})_{il} \tilde x^l$

$$\overset{\S 4.8.4}{=} (\Lambda^T g)_{il} \tilde x^l = \Lambda^T{}_i{}^j g_{jl} \tilde x^l = \Lambda^T{}_i{}^j \tilde x_j$$

$$\Rightarrow \tilde t^{ij\dots} \tilde x_i \tilde x_j \dots = \underline{t^{ij\dots} \Lambda^T{}_i{}^k \Lambda^T{}_j{}^l \dots \tilde x_k \tilde x_l \dots}$$

$$\underline{\tilde t^{kl\dots} \tilde x_k \tilde x_l \dots}$$

$$\Rightarrow \underline{\tilde t}^{kl\dots} = \Lambda^T{}_i{}^k \Lambda^T{}_j{}^l \dots t^{ij\dots}$$

$$= \underline{\Lambda^k{}_i \Lambda^l{}_j \dots t^{ij\dots}}$$

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the same for covariant tensors

remark: (3) Any tensor transforms like this under normal coordinate transfos; objects that do not transform like this are not tensors.

(3') In Physics one often defines tensors by this transformation property with no reference to multilinear forms.

Now we take $\mathcal{E}$ on a transformed basis $\{\tilde{e}^1, \tilde{e}^2, \tilde{e}^3\}$.

§4.8.2 def. 2 $\rightsquigarrow$ $\tilde{e}^i = e^j (\Lambda^{-1})_j{}^i$

$\rightsquigarrow$ the k-component of $\tilde{e}^i$ is $(\tilde{e}^i)_k = \underbrace{(e^j)_k}_{=\delta^j{}_k} (\Lambda^{-1})_j{}^i = (\Lambda^{-1})_k{}^i$

$= \delta^j{}_k$ since $\{e^i\}$ is cartesian

Consider

$(\Lambda^{-1} e^i)_k = (\Lambda^{-1})_k{}^j \underbrace{(e^i)_j}_{=\delta^i{}_j} = (\Lambda^{-1})_k{}^i$ $\rightsquigarrow$ $\underline{\tilde{e}^i = \Lambda^{-1} e^i}$

example: (1) A vector aka rank-1 tensor: §4.8.2 prop.(3) $\to$

$$\tilde{x}^i = D^i{}_j x^j \quad ✓$$

(2) The metric tensor aka Kronecker tensor: §4.8.2 prop.4 $\to$

$$\tilde{g} = (D^{-1})^T g D^{-1}$$

$$\to \underline{\tilde{g}^{ij}} \overset{\tilde g^{-1}=\tilde g}{=} (\tilde{g}^{-1})^{ij} = (D g^{-1} D^T)^{ij} \overset{g^{-1}=g}{=} D^i{}_k g^{kl} (D^T)_l{}^j = \underline{D^i{}_k D^j{}_l g^{kl}}$$

remark: (4) The metric tensor is special in the sense that it is invariant under normal coordinate transfos (due to $D^T g D = g$ for normal D), but (*) still holds.

example: (3) The Levi-Civita tensor $(\varepsilon_L)^{ijk} = \varepsilon(e^i, e^j, e^k)$

with $\varepsilon(x,y,z)$ the completely antisymmetric form with the property $\varepsilon(e^1,e^2,e^3) = +1$ for some right-handed cartesian basis $\{e^1,e^2,e^3\}$. Its general elements in that basis are $\varepsilon(e^i,e^j,e^k) = \varepsilon^{ijk} = \mathrm{sgn}\,\sigma\begin{pmatrix} i & j & k \\ 1 & 2 & 3 \end{pmatrix}$, then

§4.3 ex.(1

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But for any antisymmetric multilinear form f,

$$f(Dx, Dy, \dots) = (\det D)\, f(x,y,\dots)$$

(see, e.g., van der Waerden ch 4.7)

$$\to \underline{(\tilde\varepsilon_L)^{ijk}} = \varepsilon(\tilde e^i, \tilde e^j, \tilde e^k) = \varepsilon(D^{-1}e^i, D^{-1}e^j, D^{-1}e^k)$$

$$= \det(D^{-1})\, \varepsilon^{ijk} = \underline{(\det D)\, \varepsilon^{ijk}}$$

But $\underline{D^l{}_i D^m{}_j D^n{}_k\, \varepsilon^{ijk}} = \sum_\sigma \mathrm{sgn}\,\sigma\, D^l{}_{\sigma(1)} D^m{}_{\sigma(2)} D^n{}_{\sigma(3)}$

$$= \begin{vmatrix} D^l{}_1 & D^l{}_2 & D^l{}_3 \\ D^m{}_1 & D^m{}_2 & D^m{}_3 \\ D^n{}_1 & D^n{}_2 & D^n{}_3 \end{vmatrix} = \mathrm{sgn}\,\sigma\begin{pmatrix} l & m & n \\ 1 & 2 & 3 \end{pmatrix} \det D = (\det D)\, \varepsilon^{lmn} = \underline{(\tilde\varepsilon_L)^{lmn}}$$

remark: (5) A rank-N tensor can be considered a set of 3^N scalars $t^{i_1 \cdots i_N}$ that are associated with a coordinate system and obey the transformation rule (*).

(6) If we assign 3^N scalars to each coordinate system, they may or may not form a tensor.

def. 2: Assign the object $\varepsilon^{ijk} = \operatorname{sgn} \sigma \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$ from ex (1) to every normal coordinate system, not just to the right-handed cartesian one. This object we call the Levi-Civita symbol (as opposed to the L-C tensor).

remark: (7) By definition, ε^{ijk} is invariant under coordinate transfo:

$$\boxed{\tilde{\varepsilon}^{ijk} = \varepsilon^{ijk}}$$

(8) The Levi-Civita tensor $(\varepsilon_L)^{ijk}$ is equal to the Levi-Civita symbol only for a particular right-handed cartesian coordinate system.

(9) ε^{ijk} is not a tensor:

$$D^i{}_l \, D^j{}_m \, D^k{}_n \, \varepsilon^{lmn} \overset{ex(3)}{=} (\det D)\, \varepsilon^{ijk} = (\det D)\, \tilde{\varepsilon}^{ijk}$$

$$(\det D)^2 = 1 \rightsquigarrow \tilde{\varepsilon}^{ijk} = (\det D)\, D^i{}_l \, D^j{}_m \, D^k{}_n \, \varepsilon^{lmn}$$

which is different from (*)

def. 3: A field $t^{i_1 \cdots i_N}(x)$ that transforms as

$$\boxed{\hat{t}^{i_1 \cdots i_N}(\hat{x}) = (\det D)\, D^{i_1}{}_{j_1} \cdots D^{i_N}{}_{j_N} \, t^{j_1 \cdots j_N}(x)}$$

is called a pseudotensor field of rank N.

remark: (10) The Levi-Civita symbol is a pseudotensor of rank 3.

5.2 Gradient, curl, divergence

Let $f(x)$ be a scalar-valued fct. on V and let $t_i(x) := \frac{\partial}{\partial x^i} f(x)$ be its partial derivative with respect to the coordinate x^i.

Perform a coordinate transf. from CS to $\widehat{CS}$:

$$\underline{\hat{t}_i(\hat{x})} = \frac{\partial}{\partial \hat{x}^i} \tilde{f}(\hat{x}) = \frac{\partial}{\partial \hat{x}^i} f(x) \quad \text{since } \tilde{f}(\hat{x}) = f(x) \text{ since } f \text{ is scalar.}$$

$$= \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial \hat{x}^i} = \underline{t_j(x) (\Delta^{-1})^j{}_i} = \underline{\left((\Delta^{-1})^T \right)_i{}^j \, t_j(x)} \quad (*)$$

proposition 1: The gradient of a scalar field, i.e., the partial derivatives with respect to the contravariant coordinates transforms as a covariant vector.

proof: $(*)$ plus Problem ~~#~~ 25

remark: (1) One often writes $\partial_i f(x) := \frac{\partial}{\partial x^i} f(x)$

(2) Analogously, $\partial^i f(x) = \frac{\partial}{\partial x_i} f(x)$ is a contravariant vector field.

Now consider the curl of a vector field $v^i(x)$, defined by

$c^i(x) \equiv (\vec{\nabla} \times v)^i(x) := \epsilon^{ijk} \partial_j v_k(x)$ and transform to $\widehat{CS}$:

proposition 2: The curl of a vector field transforms as a pseudovector.

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proof: Problem 26

Finally, consider

$$d(x) \equiv \operatorname{div} v(x) := \partial_i v^i(x)$$ and transform to CS:

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proposition 3: The divergence of a vector field transforms as a scalar

proof: Problem 26

5.3 Tensor products, and tensor traces

Generalize the tensor product of §4.4 def (4):

def. 1: Let s, t be tensors rank N and M, respectively. Then

$$u = s \otimes t$$

is defined by

$$u^{i_1 \dots i_{N+M}} := s^{i_1 \dots i_N} t^{i_{N+1} \dots i_{N+M}}$$

and is called the tensor product of s and t.

proposition 1: u is a tensor of rank $N+M$ if s and t are both tensors or both pseudotensors, and a pseudotensor of rank $N+M$ otherwise.

Problem 27

Proof

proof: Problem 27

def. 2: Let $t^{i_1 \dots i_{N+2}}$ be a (pseudo) tensor of rank $N+2$. Then the (1,2)-trace or contraction $u^{k_1 \dots k_N}$ of t is defined as

$$u^{k_1 \dots k_N} := g_{ij}\, t^{ij k_1 \dots k_N} = t_i{}^{i k_1 \dots k_N}$$

proposition 2: u is a (pseudo)tensor of rank N

proof: Problem 27

remark: (1) The curl, $c^i(x) = \epsilon^{ijk}\partial_j v_k(x)$ can be considered as the contraction of the rank-5 pseudotensor $\epsilon^{ijk}\partial^l v^m$. It thus must be a pseudovector, in agreement with §5.2

5.4 Minkowski tensors

Consider M_4, i.e., $\mathbb{R}^4$ with metric $g = (+,-,-,-)$.

Let A be a vector in M_4 with contravariant coordinates

$$A^\mu \quad (\mu = 0,1,2,3) \quad \text{or} \quad A^\mu = (A^0, \vec{A}) \text{ with } \vec{A} = (A^1, A^2, A^3)$$

and covariant coordinates

$$A_\mu = (A^0, -\vec{A})$$

remark: (1) $\vec{A}$ can be considered a Euclidean vector in the subspace spanned by the basis vectors $e_{1,2,3}$ of M_4.

Consider a rank-2 tensor

$$F^{\mu\nu} = \left(\begin{array}{c|ccc} F^{00} & F^{01} & F^{02} & F^{03} \\ \hline F^{10} & F^{11} & F^{12} & F^{13} \\ F^{20} & F^{21} & F^{22} & F^{23} \\ F^{30} & F^{31} & F^{32} & F^{33} \end{array}\right) = \left(\begin{array}{c|c} F^{00} & \vec{F}_h \\ \hline \vec{F}_v & F^{ij} \end{array}\right)$$

where $\vec{F}_h$ and $\vec{F}_v$ can be considered vectors in the Euclidean subspace, and F^{ij} can be considered a Euclidean rank-2 tensor

remark: (2) If $F^{\mu\nu} = F^{\nu\mu}$ (symmetric tensor), then $\vec{F}_v = \vec{F}_h$

If $F^{\mu\nu} = -F^{\nu\mu}$ (antisymmetric tensor), then $\vec{F}_v = -\vec{F}_h$ and F^{ij} is antisymmetric.

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lemma: Antisymmetric Euclidean rank-2 tensors are isomorphic to Euclidean pseudovectors.

proof: $t^{ij} = -t^{ji}$ $\rightsquigarrow$ $t = \begin{pmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{pmatrix}$

$\rightsquigarrow$ $t^{ij} = \varepsilon^{ijk} v_k$

t is a tensor, ε is a pseudotensor $\rightsquigarrow$ v is a pseudovector

proposition: Any antisymmetric rank-2 Minkowski tensor can be written

$$F^{\mu\nu} = \left(\begin{array}{c|c} 0 & \vec{e} \\ \hline -\vec{e} & t^{ij} \end{array}\right) = \left(\begin{array}{c|ccc} 0 & & \vec{e} & \\ \hline & 0 & v_3 & -v_2 \\ -\vec{e} & -v_3 & 0 & v_1 \\ & v_2 & -v_1 & 0 \end{array}\right)$$

with $\vec{e}$ a Euclidean vector and $\vec{v}$ a Euclidean pseudovector

remark: (3) $F^{\mu}{}_{\nu} = F^{\mu\lambda} g_{\lambda\nu} = \left(\begin{array}{c|c} 0 & \vec{e} \\ \hline -\vec{e} & t^{ij} \end{array}\right) \left(\begin{array}{c|c} + & \\ \hline & - \; - \; - \end{array}\right) = \left(\begin{array}{c|c} 0 & -\vec{e} \\ \hline -\vec{e} & -t^{ij} \end{array}\right)$

etc.

(4) $F_{\mu\nu} F^{\mu\nu} = 2(\vec{v}^2 - \vec{e}^2)$ is a Minkowski scalar

(a contraction of $F^{\kappa\lambda} F^{\mu\nu}$).