Problem Assignment # 2 10/08/2020
due 10/15/2020

1.1.5 **Equivalence relations**

Consider a relation $\sim$ on a set X as in ch. 1 §1.3 def. 1, but with the properties

i) $x \sim x \quad \forall x \in X$ (reflexivity)

ii) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$ (symmetry)

iii) $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$ (transitivity)

Such a relation is called an *equivalence relation.* Which of the following are equivalence relations?

a) n divides m on $\mathbb{N}$.

b) $x \leq y$ on $\mathbb{R}$.

c) g is perpendicular to h on the set of straight lines $\{g, h, \ldots\}$ in the cartesian plane.

d) a equals b modulo n on $\mathbb{Z}$, with $n \in \mathbb{N}$ fixed.

hint: "a equals b modulo n", or $a = b \bmod (n)$, with $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$, is defined to be true if $a - b$ is divisible on $\mathbb{Z}$ by n; i.e., if $(a - b)/n \in \mathbb{Z}$.

(3 points)

1.1.6 **Bounds for $n!$**

Prove by mathematical induction that

$$n^n/3^n < n! < n^n/2^n \quad \forall\, n \geq 6$$

hint: $(1 + 1/n)^n$ is a monotonically increasing function of n that approaches Euler's number e for $n \to \infty$.

(4 points)

1.1.7 **All ducks are the same color**

Find the flaw in the "proof" of the following

proposition: All ducks are the same color.

proof: $n = 1$: There is only one duck, so there is only one color.

$n = m$: The set of ducks is one-to-one correspondent to $\{1, 2, \ldots, m\}$, and we assume that all m ducks are the same color.

$n = m+1$: Now we have $\{1, 2, \ldots, m, m+1\}$. Consider the subsets $\{1, 2, \ldots, m\}$ and $\{2, \ldots, m, m+1\}$. Each of these represent sets of m ducks, which are all the same color by the induction assumption. But this means that ducks #2 through m are all the same color, and ducks #1 and $m+1$ are the same color as, e.g., duck #2, and hence all ducks are the same color.

remark: This demonstration of the pitfalls of inductive reasoning is due to George Pólya (1888 - 1985), who used horses instead of ducks.

(2 points)

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1.2.1 **Pauli group**

The Pauli matrices are complex 2×2 matrices defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

Now consider the set P_1 that consists of the Pauli matrices and their products with the factors -1 and $\pm i$:

$$P_1 = \{\pm\sigma_0, \pm i\sigma_0, \pm\sigma_1, \pm i\sigma_1, \pm\sigma_2, \pm i\sigma_2, \pm\sigma_3, \pm i\sigma_3\}$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)