PHYS 610 Mathematical Methods for Scientists S 2020

Problem Assignment # 5 10/29/2020
due 11/05/2020

1.4.5. $\mathbb{R}$ **as a metric space**

Consider the reals $\mathbb{R}$ with $\rho : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\rho(x, y) = |x - y|$. Show that this definition makes $\mathbb{R}$ a metric space.

(3 points)

1.4.6. **Limits of sequences**

a) Show that a sequence in a metric space has at most one limit.

hint: Assume there are two limits, and use the triangle inequality to show that they must be the same.

b) Show that every sequency with a limit is a Cauchy sequence.

(3 points)

1.4.7. **Banach space**

Let B be a K-vector space (k = $\mathbb{R}$ or $\mathbb{C}$) with null vector θ. Let $||\ldots|| \ : \ \mathrm{B} \to \mathbb{R}$ be a mapping such that

(i) $||x|| \geq 0 \ \forall \ x \in \mathrm{B}$, and $||x|| = 0$ iff $x = \theta$.

(ii) $||x + y|| \leq ||x|| + ||y|| \ \forall \ x, y \in \mathrm{B}$.

(iii) $||\lambda x|| = |\lambda| \cdot ||x|| \ \forall \ x \in \mathrm{B}, \lambda \in \mathrm{K}$.

Then we call $||\ldots||$ a **norm** on B, and $||x||$ the **norm** of x.

Further define a mapping $d \ : \ \mathrm{B} \times \mathrm{B} \to \mathbb{R}$ by

$d(x, y) := ||x - y|| \ \forall \ x, y \in \mathrm{B}$

Then we call $d(x, y)$ the **distance** between x and y.

a) Show that d is a metric in the sense of §4.5, i.e., that every linear space with a norm is in particular a metric space.

If the normed linear space B with distance/metric d is complete, then we call B a **Banach space** or **B-space**.

b) Show that $\mathbb{R}$ and $\mathbb{C}$, with suitably defined norms, are B-spaces. (For the completeness of $\mathbb{R}$ you can refer to §4.5 example (3), and you don't have to prove the completeness of $\mathbb{C}$ unless you insist.)

Now let B^* be the dual space of B, i.e., the space of linear functionals ℓ on B, and define a norm of ℓ by

$||\ell|| := \sup_{||x||=1} \{|\ell(x)|\}$

c) Show that the such defined norm on B^* is a norm in the sense of the norm defined on B above.

(In case you're wondering: B^* is complete, and hence a B-space, but the proof of completeness is difficult.)

(5 points)

1.4.8. **Hilbert space**

a) Show that the norm on a Hilbert space defined by §4.7 def. 1 is a norm in the sense of the definition in Problem 1.4.7.

hint: Use the Cauchy-Schwarz inequality (§4.7 lemma).

b) Show that the mappings ℓ defined in §4.7 def. 4 are linear forms in the sense of §4.3 def. 1(a).

(3 points)