Problem Assignment # 6 11/05/2020
due 11/12/2020

1.4.9. **Lorentz transformations in M_2**

Consider the 2-dimensional Minkowski space M_2 with metric $g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and 2×2 matrix representations of the pseudo-orthogonal group $O(1,1)$ that leaves g invariant.

a) Let $\sigma, \tau = \pm 1$, and $\phi \in \mathbb{R}$. Show that any element of $O(1,1)$ can be written in the form

$$D_{\sigma,\tau}(\phi) = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix} \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}$$

To study $O(1,1)$ it thus suffices to study the matrices $D(\phi) := D_{+1,+1} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$.

b) Show explicitly that the set $\{D(\phi)\}$ forms a group under matrix multiplication (which is a subgroup of $O(1,1)$ that is sometimes denoted by $SO^+(1,1)$), and that the mapping $\phi \to D(\phi)$ defines an isomorphism between this group and the group of real numbers under addition.

c) Show that there exists a matrix J (called the *generator* of the subgroup) such that every $D(\phi)$ can be written in the form

$$D(\phi) = e^{J\phi}$$

and determine J explicitly.

(6 points)

1.4.10. **Time-like and space-like intervals**

Consider two points (ct_x, x^1, x^2, x^3) and (ct_y, y^1, y^2, y^3) in Minkowski space. The interval between the two points is called *time-like* if

$$c^2(t_x - t_y)^2 > (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad ,$$

and *space-like* if

$$c^2(t_x - t_y)^2 < (x^1 - y^1)^2 + (x^2 - y^2)^2 + (x^3 - y^3)^2 \quad .$$

Show that in interval that is time-like or space-like in some inertial frame is also time-like or space-like in any other inertial frame. (This reflects the invariance of the speed of light.)

(2 points)

. . . **/over**

1.4.11. **Special Lorentz transformations in** M_4

Consider the Minkowski space M_4.

a) Show that the following transformations are Lorentz transformations:

i) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ 0 & R^i{}_j \end{pmatrix} \equiv R^\mu{}_\nu$ (rotations)

where $R^i{}_j$ is any Euclidian orthogonal transformation.

ii) $D^\mu{}_\nu = \begin{pmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv B^\mu{}_\nu$ (Lorentz boost along the x-direction)

with $\alpha \in \mathbb{R}$.

iii) $D^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv P^\mu{}_\nu$ (parity)

iv) $D^\mu{}_\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv T^\mu{}_\nu$ (time reversal)

b) Let L be the group of all Lorentz transformations. Show that the rotations defined in part a) i) are a subgroup of L, and so are the Lorentz boosts defined in part a) ii).

c) Let $I^\mu{}_\nu = \delta^\mu{}_\nu$ be the identity transformation. Show that the sets $\{I, P\}$, $\{I, T\}$, and $\{I, P, T, PT\}$ are subgroups of L.

(4 points)

1.4.12. **General Lorentz transformations in** M_4

Let D be a general Lorentz transformation in M_4.

a) Show that $|D^0{}_0| \geq 1$, and that $(D^0{}_1)^2 + (D^0{}_2)^2 + (D^0{}_3)^2 = (D^1{}_0)^2 + (D^2{}_0)^2 + (D^3{}_0)^2$.

b) Let $L_{++} = \{D \in L; \det D > 0, D^0{}_0 > 0\}$. (This is called the set of proper orthochronous Lorentz transformations, and one can show that it is a subgroup of L.) Show that any Lorentz transformation can be written as an element of L_{++} followed by either P, or T, or PT. It thus suffices to study L_{++}.

c) Show that any element of L_{++} can be written as a spatial rotation $R(\Phi, \Theta, \Psi)$ followed by a Lorentz boost $B(\alpha)$ followed by a rotation about the 3-axes followed by a rotation about the 2-axis. In a symbolic notation:

$$D = \begin{pmatrix} 1 & 0 \\ 0 & R_2(\phi)R_3(\theta) \end{pmatrix} B(\alpha) \begin{pmatrix} 1 & 0 \\ 0 & R(\Phi, \Theta, \Psi) \end{pmatrix}$$

L_{++} is thus characterized by six parameters: 3 Euler angles Φ, Θ, Ψ, the boost parameter α, and two additional rotation angles ϕ and θ.

(7 points)