PHYS 610 Mathematical Methods for Scientists F 2020

Problem Assignments # 8 11/19/2020
due 11/26/2020

2.2.2. **Another causal function**

The function considered in Problem 2.2.1 is an example of a class of complex functions called *causal functions* that are important for the theory of many-particle systems. Another member of this class is

$$g(z) = \sqrt{z^2 - 1} - z$$

Determine the spectrum and the reactive part of $g(z)$, and plot them for $-3 < \omega < 3$.

(3 points)

2.2.3. **Proof of the Cauchy-Riemann Theorem**

Prove the Cauchy-Riemann theorem from ch.2 §2.2:

a) Let $f(z) = f'(z', z'') + if''(z', z'')$ be analytic everywhere in $\Omega \subseteq \mathbb{C}$. Show that the Cauchy-Riemann equations

$$\frac{\partial f'}{\partial z'} = \frac{\partial f''}{\partial z''} \qquad \text{and} \qquad \frac{\partial f'}{\partial z''} = -\frac{\partial f''}{\partial z'}$$

hold $\forall z \in \Omega$.

hint: Start with the difference quotient $(f(z) - f(z_0))/(z - z_0)$ and require that it's limit for $z \to z_0$ exists if z_0 is approached on paths either parallel to the real axis, or parallel to the imaginary axis.

b) Let the Cauchy-Riemann equations hold in a point $z_0 \in \Omega$. Show that this implies that f is analytic in the point z_0.

hint: Consider $f(z) - f(z_0)$ and expand $f'(z', z'')$ and $f''(z', z'')$ in Taylor series about z_0.

(8 points)

2.2.4. **Exponentials**

Consider the exponential function

$$f(z) = e^z = e^{z' + iz''}$$

a) Show that $f(z)$ is analytic everywhere in $\mathbb{C}$.

b) Convince your self explicitly that the real and imaginary parts of f obey Laplace's differential equation.

c) Show that $df/dz|_z = f(z)$.

d) Show that $\cos z$ and $\sin z$, defined by

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \quad , \quad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

are analytic everywhere in $\mathbb{C}$, and that

$$\frac{d}{dz}\cos z = -\sin z \quad , \quad \frac{d}{dz}\sin z = \cos z \; .$$

(4 points)

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2.3.1. **Laurent series**

Find the Laurent series for the function

$$f(z) = 1/(z^2 + 1)$$

in the point $z = i$. That is, find the coefficients f_n that enter the theorem in ch. 2 §3.2.

(3 points)