PHYS 610 Mathematical Methods for Scientists F 2020

Problem Assignment # 2 10/08/2020
due 10/15/2020

1.1.5 **Equivalence relations**

Consider a relation $\sim$ on a set X as in ch. 1 §1.3 def. 1, but with the properties

i) $x \sim x \quad \forall x \in X$ (reflexivity)

ii) $x \sim y \Rightarrow y \sim x \quad \forall x, y \in X$ (symmetry)

iii) $(x \sim y \wedge y \sim z) \Rightarrow x \sim z$ (transitivity)

Such a relation is called an *equivalence relation*. Which of the following are equivalence relations?

a) n divides m on $\mathbb{N}$.

b) $x \leq y$ on $\mathbb{R}$.

c) g is perpendicular to h on the set of straight lines $\{g, h, \ldots\}$ in the cartesian plane.

d) a equals b modulo n on $\mathbb{Z}$, with $n \in \mathbb{N}$ fixed.

hint: "a equals b modulo n", or $a = b \bmod(n)$, with $a, b \in \mathbb{Z}$, $n \in \mathbb{N}$, is defined to be true if $a - b$ is divisible on $\mathbb{Z}$ by n; i.e., if $(a - b)/n \in \mathbb{Z}$.

(3 points)

1.1.6 **Bounds for $n!$**

Prove by mathematical induction that

$$n^n/3^n < n! < n^n/2^n \quad \forall\, n \geq 6$$

hint: $(1 + 1/n)^n$ is a monotonically increasing function of n that approaches Euler's number e for $n \to \infty$.

(4 points)

1.1.7 **All ducks are the same color**

Find the flaw in the "proof" of the following

proposition: All ducks are the same color.

proof: $n = 1$: There is only one duck, so there is only one color.

$n = m$: The set of ducks is one-to-one correspondent to $\{1, 2, \ldots, m\}$, and we assume that all m ducks are the same color.

$n = m+1$: Now we have $\{1, 2, \ldots, m, m+1\}$. Consider the subsets $\{1, 2, \ldots, m\}$ and $\{2, \ldots, m, m+1\}$. Each of these represent sets of m ducks, which are all the same color by the induction assumption. But this means that ducks #2 through m are all the same color, and ducks #1 and $m+1$ are the same color as, e.g., duck #2, and hence all ducks are the same color.

remark: This demonstration of the pitfalls of inductive reasoning is due to George Pólya (1888 - 1985), who used horses instead of ducks.

(2 points)

. . . **/over**

1.2.1 **Pauli group**

The Pauli matrices are complex 2×2 matrices defined as

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad ,$$

Now consider the set P_1 that consists of the Pauli matrices and their products with the factors -1 and $\pm i$:

$$P_1 = \{\pm\sigma_0, \pm i\sigma_0, \pm\sigma_1, \pm i\sigma_1, \pm\sigma_2, \pm i\sigma_2, \pm\sigma_3, \pm i\sigma_3\}$$

Show that this set of 16 elements forms a (nonabelian) group under matrix multiplication called the Pauli group. It plays an important role in quantum information theory.

(3 points)

1.1.5) a) No, since it is not symmetric. E.g., $2 \sim 4$, but $4 \nsim 2$.

b) No, since it is not symmetric. E.g., $2 \leq 4$ but $4 \nleq 2$

c) No, since it is not reflexive: No line is perpendicular to itself.

①

d) yes.

proof: (i) $a - a = 0$ is divisible by n $\rightsquigarrow$ $\underline{a \equiv a \bmod (n)}$

(ii) let $\underline{a \equiv b \bmod (n)}$ $\rightsquigarrow$ $\exists\, k \in \mathbb{Z}:\ a - b = kn$

$\rightsquigarrow$ $b - a = (-k)n$ $\rightsquigarrow$ $b - a$ is divisible by n

$\rightsquigarrow$ $\underline{b \equiv a \bmod (n)}$

(iii) let $\underline{a \equiv b \bmod (n)}$ and $\underline{b \equiv c \bmod (n)}$

$\rightsquigarrow$ $\exists\, k, l \in \mathbb{Z}:$ $a - b = kn$ and $b - c = ln$

$\rightsquigarrow$ $a - c = (a - b) + (b - c) = kn + ln = (k + l)n$ with $k + l \in \mathbb{Z}$

$\rightsquigarrow$ $\underline{a \equiv c \bmod (n)}$

$\rightsquigarrow$ $\underline{\underline{a \equiv b \bmod (n)}}$ is an $\underline{\underline{\text{equivalence relation}}}$ on $\mathbb{Z}$. $\square$

②

1.1.6) First prove $n^n/3^n < n!$ $\forall\, n \geq 6$

$\underline{n=6}$: $6^6/3^6 = 2^6 = 64 < 720 = 6!$ ✓

Assume $m^m/3^m < m!$

Then
$$\frac{(m+1)^{m+1}}{3^{m+1}} = \frac{m^m}{3^m}\,\frac{1}{3}\left(1+1/m\right)^m (m+1) < \frac{m^m}{3^m}\,\frac{e}{3}(m+1)$$
$$< \frac{m^m}{3^m}(m+1) \underline{<} m!\,(m+1) = \underline{(m+1)!}$$

2

Now prove $n^n/2^n > n!$ $\forall\, n \geq 6$

$\underline{n=6}$: $6^6/2^6 = 3^6 = 729 > 720 = 6!$ ✓

Assume $m^m/2^m > m!$

Then
$$\frac{(m+1)^{m+1}}{2^{m+1}} = \frac{m^m}{2^m}\,\underbrace{\frac{(1+1/m)^m}{2}}_{\geq 1}(m+1) \geq \frac{m^m}{2^m}(m+1) \underline{>} m!\,(m+1)$$
$$= \underline{(m+1)!}$$

$$\Rightarrow \quad \frac{n^n}{3^n} < n! < \frac{n^n}{2^n} \quad \forall\, n \geq 6 \qquad \square$$

2

1.1.7.) The problem lies with $n=2$.

The induction step from $n=m$ to $n=m+1$ relies on the fact that the subsets $\{1,2,\dots,m\}$ and $\{2,3,\dots,m+1\}$ have common elements. But for $n=2$, we have $m=1$, and then the two sets are $\{1\}$ and $\{2\}$, which have <u>no</u> common element!

$\leadsto$ In order for the proof to be valid, one has to prove that any <u>two</u> ducks have the same color, which is not possible.

2

1.2.1.) The Pauli matrices obey

	σ_0	σ_1	σ_2	σ_3
σ_0	σ_0	σ_1	σ_2	σ_3
σ_1	σ_1	σ_0	$i\sigma_3$	$-i\sigma_2$
σ_2	σ_2	$-i\sigma_3$	σ_0	$i\sigma_1$
σ_3	σ_3	$i\sigma_2$	$-i\sigma_1$	σ_0

i.e., $\sigma_i \sigma_j$ equals either some σ_k or some σ_k times $\pm i$

1

Now consider P_1 :

	σ_0	$-\sigma_0$	$i\sigma_0$	$-i\sigma_0$	σ_1	$-\sigma_1$	$i\sigma_1$	$-i\sigma_1$	σ_2	$-\sigma_2$	$i\sigma_2$	$-i\sigma_2$	σ_3	$-\sigma_3$	$i\sigma_3$	$-i\sigma_3$
σ_0	σ_0	$-\sigma_0$	$i\sigma_0$	$-i\sigma_0$	σ_1	$-\sigma_1$	$i\sigma_1$	$-i\sigma_1$	σ_2	$-\sigma_2$						
$-\sigma_0$	$-\sigma_0$	σ_0	$-i\sigma_0$	$i\sigma_0$	$-\sigma_1$	σ_1	$-i\sigma_1$	$i\sigma_1$	$-\sigma_2$	σ_2						
$i\sigma_0$	$i\sigma_0$	$-i\sigma_0$	$-\sigma_0$	σ_0	$i\sigma_1$	$-i\sigma_1$	$-\sigma_1$	σ_1	$i\sigma_2$	$-i\sigma_2$						
$-i\sigma_0$	$-i\sigma_0$	$i\sigma_0$	σ_0	$-\sigma_0$	$-i\sigma_1$	$i\sigma_1$	σ_1	$-\sigma_1$	$-i\sigma_2$	$i\sigma_2$						
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etc. Without even completing the table we see that

(i) The set is closed under matrix multiplication, since $\sigma_i \sigma_j$ is always some σ_k times 1 or $\pm i$.

1

(ii) Matrix multiplication is associative.

(iii) σ_0 is the unit element

(iv) Each element has an inverse, viz

$\sigma_0\sigma_0 = \sigma_0$	$\sigma_1\sigma_1 = \sigma_0$	same for
$(-\sigma_0)(-\sigma_0) = \sigma_0$	$(-\sigma_1)(-\sigma_1) = \sigma_0$	σ_2, σ_3
$(i\sigma_0)(-i\sigma_0) = \sigma_0$	$(i\sigma_1)(-i\sigma_1) = \sigma_0$	
$(-i\sigma_0)(i\sigma_0) = \sigma_0$	$(-i\sigma_1)(i\sigma_1) = \sigma_0$	

1